

THEORIES OF FIGURES OF CELESTIAL BODIES

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With a Foreword by

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FOREWORD

Sir Arthur Eddington once remarked that one of the most profound mysteries of the universe consists in the fact that everything in it rotates. The meteors, asteroids, the planets and satellites, the Sun, the stars, the clusters, and nebulae, and even the galaxies spin around their axes. As they do so their shapes are distorted—imperceptibly in the case of the Sun with its equatorial rotational velocity of but 2 km./sec., but easily distinguishable, with a small telescope, in the case of Jupiter or Saturn.

There are many hot stars whose equatorial rotational velocities approach values of the order of 500 km./sec. In a few white dwarfs even larger velocities may be found. Such stars must be greatly flattened, and in a few of them we actually observe the outflow of gases at the sharp edges of their equators, producing flat, extended gaseous rings in their equatorial planes.

Professor Jardetzky's book deals with the figures of equilibrium and distortions of rotating bodies, and it comes at a very opportune time. This theory is usually associated with the names of Poincaré, Darwin, Liapounov, and Jeans, although many other distinguished astronomers and mathematicians—among them Dr. Jardetzky himself—have made contributions to it. In its early stages the aim of the theoreticians was to trace the consequences of an ever increasing angular rate of rotation. This problem has assumed new importance in stellar astronomy because it is now possible to trace, largely from the work of L. G. Henyey in Berkeley and of B. Strömgren at the University of Chicago, how a diffuse gaseous mass condenses through the gravitational process known as Kelvin contraction, and becomes after a lapse of some hundreds of thousands of years, a main-sequence star. The angular momentum of the original cloud is preserved in the contraction, and

the newly-formed star rotates around its axis. But even before the star reaches the main sequence nuclear processes start operating in its interior, and they maintain the star for a long time—millions, billions, or even tens of billions of years—in equilibrium on the main sequence. Gradually, however, the hydrogen nuclear fuel begins to be nearly exhausted. The star then starts expanding again—at first slowly, by a process first investigated by S. Chandrasekhar and M. Schönberg, and later more and more rapidly according to a new theory by M. Schwarzschild and A. Sandage. The star blows up and cools down: it becomes a diffuse giant. Its surface rotation must change again, and accordingly also the shape of its surface. The observations show that many of these “old” giants rotate much faster than had been expected; values of the order of 75–100 km./sec. are not unusual. It should be possible to infer from these velocities the distribution of the material inside such giant stars.

Professor Jardetzky was the last of many pupils of the late A. M. Liapounov whose work he has treated in this volume. In 1918, during the Russian revolution, Liapounov settled in Odessa where Jardetzky was then a young mathematician. In the last year of his life (he committed suicide in 1918) Liapounov gave a series of lectures at the university about his investigations, which have since been published by the Soviet Academy of Sciences, in 1925, under the title *Sur certaines séries de figures d'équilibre d'un liquide hétérogène en rotation*. In this memoir he gave the most general results in the field in which his Master's thesis, in 1884, on the stability of ellipsoidal forms of equilibrium of a rotating liquid won for him immediate international recognition. It should be emphasized that many of the later results of Liapounov have remained virtually unknown in the western world.

Astronomers may be interested in a few personal remarks. A.M. Liapounov was the son of the astronomer M. Liapounov, who was in charge of the Kazan Observatory around the middle of the 19th century. There the father was engaged in various astronomical observations, including a detailed study of the positions of stars associated with the Orion nebula. This work was published

by him in collaboration with Otto Struve Sr. in 1862. The son learned the fundamentals of astronomy from his father. His mathematical inclination was, however, the result of the inspiration he received from the great mathematician Tchebyshev at St. Petersburg University. The son Liapounov spent some of the most productive years of his life as professor of analytical mechanics in the University of Kharkov, where he was associated with the father of the present writer. Unfortunately, the latter never had the opportunity of studying under Liapounov. His knowledge in this field derives from one of Liapounov's earliest students, N. N. Saltykov.

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PART I

Figures of Equilibrium of a Rotating Fluid Mass

CHAPTER I

Introduction

The theory of figures of celestial bodies has been developed by a number of most outstanding mathematicians, primarily by Maclaurin, Jacobi, Poincaré, and Liapounov. Their investigations, however, have been largely limited to the case of an isolated fluid mass in uniform rotation. This approach was justified for two reasons. First, the figures of celestial bodies actually differ but little from the figures given by the solutions of this problem (e.g., a sphere, ellipsoids of Maclaurin and Jacobi, figures of Poincaré and Liapounov), and second, the existence of such figures could be demonstrated by the most rigorous methods of mathematical analysis.

Some progress may still be possible in this area, but the development of an exact theory which would take into account conditions more closely approaching reality appears more interesting and important. Variations in figures and stratification of celestial bodies which result from internal movements should be considered as the next approximation. There are variations important primarily in the investigation of stellar structure. On the other hand, there are features observed in the Earth's crust and similar features undoubtedly existing in the crust of any other solidified planet which may be also explained by the departure from conditions of equilibrium. Thus, many problems arise in which the figures of celestial bodies have to be determined under conditions different from equilibrium.

The most exact methods used in the theory of figures of equilibrium and the most important results are discussed in the first six chapters. The second part of this book deals with some other problems concerning the figures of celestial bodies. Many

of these problems still await their solution by equally rigorous methods as those developed in the theory of figures of equilibrium.

1.1. Fundamental Problem

The solution of the following problem of Hydrodynamics is being presented for the purpose of explaining the shape of the Sun and other celestial bodies. Let a fluid mass be rotating in space as a rigid body. Let us assume that it is isolated from other bodies and that its particles are subject to mutual attractions according to Newton's law of gravitation. What is the figure of this mass?

When the problem is given in this general form, not all conditions are taken into account. Nevertheless, we can make some important preliminary conclusions using the equations of motion of a fluid mass. We call z -axis the axis of rotation of the mass and take the origin of a rectangular system of axes $Oxyz$ at the mass center. Let ω be the angular velocity. The velocity \mathbf{v} of a point M is then given by the expression

$$(1.1.1) \quad \mathbf{v} = \omega \times \mathbf{r}$$

if the vector $\mathbf{r}(x, y, z)$ is equal to \mathbf{OM} . The components of acceleration at M are: $-\omega^2 x$, $-\omega^2 y$, 0. Let fU be the gravitational potential, f the constant of gravitation and κ and p the density and pressure of the fluid. We can write the equations of motion of a fluid mass either in the vector form

$$(1.1.2) \quad \frac{d^2 \mathbf{r}}{dt^2} = \text{grad } fU - \frac{1}{\kappa} \text{grad } p$$

or in the scalar form

$$(1.1.3) \quad \begin{aligned} -\omega^2 x &= f \frac{\partial U}{\partial x} - \frac{1}{\kappa} \frac{\partial p}{\partial x}, & -\omega^2 y &= f \frac{\partial U}{\partial y} - \frac{1}{\kappa} \frac{\partial p}{\partial y}, \\ 0 &= f \frac{\partial U}{\partial z} - \frac{1}{\kappa} \frac{\partial p}{\partial z} \end{aligned}$$

If we assume that ω is constant, we obtain by (1.1.3)

$$(1.1.4) \quad f dU + \frac{\omega^2}{2} d(x^2 + y^2) = \frac{1}{\kappa} dp$$

In two cases, namely, $\kappa = \text{const.}$ or $\kappa = \varphi(p)$ we have an integral of this equation in the form

$$(1.1.5) \quad fU + \frac{\omega^2 s^2}{2} = \int \frac{dp}{\kappa} + \text{const.}$$

where $s = \sqrt{x^2 + y^2}$ is the distance of a particle from the axis of rotation.

For Newton's law of gravitation we have¹

$$(1.1.6) \quad U = \int_V \frac{\kappa' dV'}{D}$$

where D is the distance of the particle at M from an element dV' .

If a fluid mass is isolated in space, the pressure at its boundary surface vanishes:

$$(1.1.7) \quad p = 0$$

Thus the boundary surface belongs to the set of surfaces of equal pressure. The general condition for this set may be immediately obtained by putting $p = \text{const.}$ in (1.1.5)

$$(1.1.8) \quad \int_V \frac{\kappa' dV'}{D} + \frac{\omega^2 s^2}{2f} = \text{const.}$$

In the case $\kappa = \varphi(p)$ the surfaces $p = \text{const.}$ coincide with the surfaces of equal density.

In general, the volume V of the fluid mass in the equation (1.1.8) is unknown and the shape of this mass must be chosen in such a way that this equation will be satisfied. Thus, the problem we have posed in its simplest form, leads to the solution of a functional equation (1.1.8), where the limits of integration in the first term are unknown.²

We obtain a particular case of the problem on considering figures of equilibrium of a fluid mass at rest. Since we assume now that

¹ The theory of figures of celestial bodies can be readily extended to the case of a law of gravitation which yields a potential differing but little from (1.1.6) as it was shown by Jurdetzky [5].

² This equation can be transformed in an integral equation by using the method of Dirichlet. The resulting equation is a nonlinear one with four-fold integrals having limits $\pm \infty$ and a very complicated integrand.

$\omega = 0$, these figures must be determined by the solutions of the equation

$$(1.1.9) \quad \int_v \frac{\kappa' dV'}{D} = \text{const.}$$

We have to keep in mind that on deriving the integral (1.1.5) we have considered fluids as having the property that either $\kappa = \text{const.}$, i.e., homogeneous and incompressible fluids, or $\kappa = \varphi(p)$, i.e., compressible fluids which are homogeneous by composition but can have a density varying with pressure. It is evident, however, that the equations (1.1.3) hold without these restrictions, and we can investigate also the figures of equilibrium of a fluid mass that is composed of a finite or even of an infinite number of incompressible fluids having different densities or figures of a compressible and inhomogeneous mass.

There are types of motion other than a uniform rotation of a fluid mass about a fixed axis which are also determined by equations (1.1.3). The problems concerning these more general movements will be discussed later.

1.2. Remarks Concerning Basic Assumptions

In order to solve a problem in the theory of figures of equilibrium or, in general, to determine the figures of a rotating fluid mass, certain basic assumptions are made concerning the physical properties of the fluid, the character of motion, the mass distribution, and others. Some of these assumptions will be discussed now.

For the motion of a fluid the condition of continuity can be written either in the form

$$(1.2.1) \quad \frac{d\kappa}{dt} + \kappa \operatorname{div} \mathbf{v} = 0$$

or

$$(1.2.1') \quad \frac{\partial \kappa}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \kappa + \kappa \operatorname{div} \mathbf{v} = 0$$

In case of a permanent motion, we have $\partial \kappa / \partial t = 0$, i.e., the density κ does not depend on time explicitly. But, in some

invariable figures of a fluid mass, the density of a particle will not vary along its path and, therefore, we have $d\kappa/dt = 0$. By (1.2.1) and (1.2.1') we obtain $\text{div } \mathbf{v} = 0$ and

$$(1.2.2) \quad \mathbf{v} \cdot \text{grad } \kappa = 0$$

Since the vector $\text{grad } \kappa$ is perpendicular to the surface of equal density $\kappa = \text{const.}$, the velocity \mathbf{v} must be in the tangential plane passing through the given particle or point in the fluid.

If the motion of a fluid mass has the characteristics just mentioned, the equation of continuity has the form

$$(1.2.3) \quad \text{div } \mathbf{v} = 0$$

for an incompressible fluid as well as for a compressible one. It is easy to see that, if the velocity \mathbf{v} is given, by the expression (1.1.1), the equation (1.2.3) will be satisfied either by $\omega = \text{const.}$ or by $\omega = F(x^2 + y^2, z)$. It is, namely,

$$\text{div } \mathbf{v} = -\frac{\partial(\omega y)}{\partial x} + \frac{\partial(\omega x)}{\partial y} = 0$$

in both cases. Thus, the assumption of a rotating figure of equilibrium is compatible with the condition (1.2.3).

The second assumption about the character of motion concerns the existence of an axis of rotation. It has been proved by Appell [1] that a necessary condition of the existence of a figure of equilibrium of a fluid mass is that the rotation occurs with respect to an axis having a direction invariable in space and coinciding with one of the principal axes of inertia of this mass. Jardeitzky [3] has shown that this condition holds also for a more general kind of rotation, namely, for the so-called zonal rotation and for mixed systems, such as a body composed of solid and fluid parts.

It is not necessary to make special assumptions about the symmetry of figures of equilibrium. It is known (see, for example, Appell [1]) that the axis of rotation can be but is not necessarily an axis of symmetry of a rotating fluid mass. It can be shown, for example, in case of a homogeneous mass that there can be a finite number of planes of symmetry passing through the axis.

On the other hand, Lichtenstein [1] has proved that a figure of equilibrium must have a plane of symmetry perpendicular to the axis of rotation.

As to the assumptions concerning the physical properties of the fluid, few remarks have to be added. We have seen that we have to discuss the figures of a homogeneous or heterogeneous, compressible or incompressible, fluid mass. It is usually assumed that the fluid is isotropic. A theory of figures of an anisotropic fluid has not yet been developed.

In the most simple case, a fluid mass is considered incompressible and homogeneous in the sense that the density has equal values everywhere and that these values cannot be changed by a varying pressure. Then, the equation (1.1.8) takes the form

$$(1.2.4) \quad \int_V \frac{dV'}{D} + \frac{\omega^2 s^2}{2f\kappa} = \text{const.}$$

The left side of this equation is an unknown function of coordinates determined by the shape of the boundary surface S of the volume V . If the value of the constant right-side member is taken for $p = 0$, the problem of figures of equilibrium is to find a set of surfaces S such that for the corresponding volume V the condition (1.2.4) will be satisfied, i.e., that (1.2.4) will be the equation of any of these surfaces.

The second problem in the theory of figures of equilibrium deals with the case where the density depends on the pressure only $\kappa = \varphi(p)$. In classical Hydrodynamics, these fluids are usually referred to as the inhomogeneous. To solve the problem in this case, we have to make use of equation (1.1.8) or (1.1.5), since the last equation yields the level surfaces. Under our assumption about the physical properties of the fluid, the surfaces $p = \text{const.}$, $\kappa = \text{const.}$, and the level surfaces (the surfaces of equal gravity potential in case of the Earth) will coincide if (1.1.5) is satisfied. In this equation, the density κ is the variable which will determine the stratification. The outer layer of the fluid corresponds to $\kappa = \varphi(0)$.

As to the equation of state $\kappa = \varphi(p)$, it can determine a con-

tinuous function, or we can consider a system of a finite number of fluids differing in their physical properties superimposed on each other. For each layer, then, a particular equation of state will hold. A fluid for which no assumption such as $\kappa = \text{const.}$ or $\kappa = \varphi(p)$ is made can be also considered. Then a continuous stratification is postulated with a density expressed in terms of coordinates. For such fluids the conditions (1.1.4) and (1.1.5) do not always hold and, therefore, the discussion must begin with the equations (1.1.3).

In problems concerning the figures of equilibrium mentioned before, the stratification of an inhomogeneous fluid is not known in advance and is to be found. The figure of such a fluid is determined by the shape of the outer layer. Of course it is not always necessary to follow this direct way. One can make assumptions about the shape of the boundary surface and postulate certain stratification. Then, it must be verified that all other conditions are not violated by the assumptions made.

Volterra [2] has proved that in case of equilibrium the layers of equal density cannot have the form of a surface of the second degree, and this proof was generalized for a zonal rotation (see, for example, Jardetzky, [4]). However, when an approximate solution is considered, for example, for a slow rotation for which we know that the surfaces of equal density will differ but little from a spherical shape, an ellipsoidal stratification presents a certain degree of approximation.

It is usually assumed that in a figure of equilibrium the density is increasing towards its centrum. For example, the mass distribution in the Earth has been represented by the well-known laws of Levy, Lipschitz, Roche, and some others of a similar type.

We shall mention one more assumption, namely, that in the case of an inhomogeneous fluid the surfaces of equal density form a set of closed surfaces each enclosing the preceding one.

1.3. Density Distribution

In the investigations concerning the figures of equilibrium, it is usually assumed that the density is an increasing function of

the distance from the surface of the body. However, for one case, at least, a proof of this fact can be given. Having demonstrated that a sphere is a unique figure of equilibrium of an isolated fluid mass at rest, Liapounov [10] had also shown that the density will increase from the surface toward the center because of the conditions of equilibrium. His proof is as follows.

Let A be the radius of a sphere which is the boundary surface of a fluid mass at rest and κ the density. It is obvious that if an inhomogeneous fluid mass is in equilibrium, it has to be composed of concentric spherical layers. In general, the density can be a function of a parameter α . However, if the fluid is compressible and $\kappa = \varphi(p)$ it may be shown that $\kappa = \psi(\alpha)$ cannot be an arbitrary function. By equation (1.1.9) we can put on a level surface

$$(1.3.1) \quad \int_V \frac{\kappa' dV'}{D} = \text{const.} = \alpha$$

It follows then by (1.2.4) and (1.1.5) that

$$(1.3.2) \quad \frac{1}{f} \int_0^p \frac{dp}{\varphi(p)} = \alpha - \alpha_0$$

if $\omega = 0$ and α_0 corresponds to $p = 0$, i.e., to the free surface of the fluid. Thus, we obtain from this equation $p = p(\alpha)$ and, on inserting this function in the equation of state, it is $\kappa = \psi(\alpha)$. In order to prove that the density is a function increasing toward the center, we shall transform the integral (1.3.1). Let U_M be the potential at a point $M(x, y, z)$ at a distance r from the center O of the fluid mass and the element $\kappa' dV'$ taken at a point $M'(x', y', z')$ at a distance r' . Taking the polar axis along OM , we assume that the angle θ corresponds to the colatitude and λ to the longitude. Let θ' be the angle between r and r' , $d\sigma'$ the element of the sphere having unit radius. Then $d\sigma' = \sin \theta' d\theta' d\lambda$ and $D = \sqrt{r^2 + r'^2 - 2rr' \cos \theta'}$. Now we can write

$$(1.3.3) \quad U_M = \int_V \frac{\kappa' dV'}{D} = \int_0^A \kappa' r'^2 dr' \int \frac{d\sigma'}{D}$$

since the volume element dV' at the distance r' is $r'^2 d\sigma' dr'$.

The integral taken over the unit sphere is

$$\begin{aligned} \int \frac{d\sigma'}{D} &= 2\pi \int_0^\pi \frac{\sin \theta' d\theta'}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}} = \frac{2\pi}{rr'} [\sqrt{r^2 + r'^2 - 2rr' \cos \theta'}]_0^\pi \\ &= \frac{2\pi}{rr'} [r + r' - |r - r'|] \end{aligned}$$

If $r' < r$ this expression is equal to

$$\int \frac{d\sigma'}{D} = \frac{4\pi}{r}$$

and if $r < r'$ we have

$$\int \frac{d\sigma'}{D} = \frac{4\pi}{r'}$$

For each point M in fluid's interior it is, then, by (1.3.3)

$$\int_V \frac{\kappa' dV'}{D} = \frac{4\pi}{r} \int_0^r \kappa' r'^2 dr' + 4\pi \int_r^A \kappa' r' dr'$$

Now, if we take all points of a sphere $r = a$, it is

$$\int_V \frac{\kappa' dV'}{D} = \frac{4\pi}{a} \int_0^a \kappa a^2 da + 4\pi \int_a^A \kappa a da$$

where κ must be expressed in terms of a .

Hence

$$(1.3.4) \quad \frac{d}{da} \int \frac{\kappa' dV'}{D} = -\frac{4\pi}{a^2} \int_0^a \kappa a^2 da$$

The integral taken with respect to the variable a is obviously a positive number because of the condition $\kappa > 0$. Since

$$\frac{d}{da} \int \frac{dp}{\varphi(p)} = \frac{d}{dp} \int \frac{dp}{\varphi(p)} \cdot \frac{dp}{da} = \frac{1}{\kappa} \frac{dp}{da}$$

we obtain by (1.1.5) and (1.3.4) that

$$(1.3.5) \quad \frac{dp}{da} = \kappa \frac{d}{da} \int \frac{\kappa' dV'}{D} < 0$$

Thus, the pressure in a spherical figure of equilibrium is

increasing toward the center. If we consider the case where the density of a fluid is an increasing function of the pressure, the density in a figure of equilibrium must also increase from the free surface toward the center.

In case of a heterogeneous and incompressible liquid we usually assume that the density is an increasing function of the depth and that otherwise the equilibrium would be unstable.

Another proof of the theorem that a sphere is a unique solution of the problem of equilibrium of a fluid mass at rest has been given by Carleman [1].

1.4. Development of Methods of Solution

We have seen that the problem of figures of equilibrium of a fluid mass requires even in its simplest form, the solution of a functional equation (1.1.8). This problem is not yet solved in its general form, i.e., all the possible figures of equilibrium are not yet known despite the fact that the most competent investigators have suggested highly ingenious methods. Using these methods, solutions were obtained for a set of particular problems. The most complete review of the theories concerning the problem of equilibrium of a fluid mass is given in Volume IV of the *Traité de Mécanique Rationnelle* by P. Appell. Only a short survey necessary for understanding the development of these theories will be presented here.

As is well known, Newton has seen the importance of the law of gravitation for the explanation of figures of celestial bodies and, on investigating the conditions of equilibrium in canals directed from the Earth's center toward the equator and the poles, reached the conclusion that the figure of the Earth is an ellipsoid. Maclaurin was the first who gave (in 1742) a rigorous mathematical proof of the fact that an ellipsoid of rotation can be a figure of equilibrium of an isolated, rotating, homogeneous fluid mass. One year later the investigation of Clairaut dealing with an inhomogeneous fluid was published. There were no exact solutions known in the 18th Century other than Maclaurin's ellipsoid, but attempts to obtain such solutions were made by

Maupertius, Simpson, Legendre, and Laplace. It was Legendre's suggestion to identify the equation of the free surface of fluid with the condition that the sum of potentials due to gravitation and to the centrifugal force is a constant on this surface.

The easiest case in the theory of figures of equilibrium was obviously that of a homogeneous fluid ($\kappa = \text{const.}$), and we shall mention first the progress made in this case. The potential of a homogeneous ellipsoid E corresponding to Newton's law of gravitation at a point $M(x, y, z)$ inside the mass can be written in the form

$$(1.4.1) \quad fU = \text{const.} - L_x \frac{x^2}{2} - L_y \frac{y^2}{2} - L_z \frac{z^2}{2}$$

Inserting this expression in (1.3.5) of (1.1.8) and taking the value of the constant term which the latter has at the surface of equal pressure passing through the point M , we obtain

$$(1.4.2) \quad (L_x - \omega^2)x^2 + (L_y - \omega^2)y^2 + L_z z^2 = C$$

This condition holds for all points of such a surface, and a particular value of the constant will determine the surface of the ellipsoid E . We shall now assume that this surface is given by the equation

$$(1.4.3) \quad \frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$$

On putting $L_x = L_y = L$ in (1.4.2) we have for an ellipsoid of revolution the equation

$$(1.4.2') \quad (L - \omega^2)(x^2 + y^2) + L_z z^2 = C$$

and, since the equations (1.4.3) and (1.4.2') must represent the same surface, we obtain two conditions

$$(1.4.4) \quad a^2(L - \omega^2) = c^2 L_z = C$$

The expressions L_x and L_y are known in terms of the axes of the ellipsoid as well as of its density.³ Therefore, we can find the corresponding values of ω and C from the conditions (1.4.4). Thus, the surface of the fluid mass will be completely determined.

³ See (2.2.7).

It may be easily proved (as shown on p. 30) that, if an ellipsoid will be a figure of equilibrium, it must be an oblate one. It may be seen from (1.4.4) and the expressions for L_x and L_y that in different conditions the ratios of axes play an important part. If we put $l = \sqrt{a^2 - c^2}/c$ a condition may be found which ω , l , and κ have to satisfy. This condition is

$$(1.4.5) \quad \frac{\omega^2}{2\pi f \kappa} = \frac{(3 + l^2) \tan^{-1} l - 3l}{l^3}$$

The discussion of this equation shows that there is a certain limit value of ω and that, in general, three cases must be considered corresponding to the condition.

$$(1.4.6) \quad h = \frac{\omega^2}{2\pi f \kappa} \begin{matrix} < \\ = \\ > \end{matrix} 0.2247 \dots$$

In the first case there are two ellipsoids of Maclaurin which correspond to the same value of ω . If h has exactly the numerical value given in (1.4.6), or is larger, only one such ellipsoid is possible. Finally, when $\omega > \omega_0$ where ω_0 is the limit value mentioned above, no ellipsoid of revolution can be a figure of equilibrium of a rotating fluid mass.

In 1834, Jacobi had shown that an ellipsoid with three different axes can also be a figure of equilibrium, if the axes satisfy certain conditions. Now, if

$$(1.4.7) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is the equation of such an ellipsoid, it must be identical to (1.4.2). Therefore, the conditions

$$(1.4.8) \quad a^2(L_x - \omega^2) = b^2(L_y - \omega^2) = c^2 L_z$$

are required. Further analysis of this case was given also by Otto Meyer, Liouville, Smith, and Plana. We will only mention here that L_x , L_y , and L_z can be expressed in terms of a , b , c and, therefore, if ω is given the ratios $s = c^2/a^2$ and $t = c^2/b^2$ are determined by (1.4.8). On making use, then, of the expression

for the given mass of a fluid $m = \frac{4}{3}\pi abc\kappa$, we can compute the axes themselves. A new limit value for the angular velocity is determined for these so-called Jacobi's ellipsoids. This value $\omega = \omega_1$, is given by the condition $h = 0.1871\dots$. When

$$(1.4.9) \quad h < 0.1871\dots$$

there is only one ellipsoid of Jacobi possible which corresponds to a given value of ω and represents a figure of equilibrium. For the limit value ω_1 of the angular velocity, Jacobi's ellipsoid becomes an ellipsoid of revolution.

For the discussion of results of the theory the angular momentum

$$(1.4.10) \quad M = I\omega$$

(where I is the moment of inertia with respect to the axis of rotation) can be used instead of the angular velocity. The initial state of a given mass can also be characterized by it. Then, the results mentioned above are as follows. (a) If $M = 0$, the angular velocity ω is equal to zero too, and the corresponding figure of equilibrium is a sphere. (b) If the initial angular momentum M has values between 0 and M_0 (or ω varies from 0 to ω_0 — the limit value), the corresponding figure of equilibrium is an ellipsoid of revolution. (c) When M belongs to the interval (M_0, ∞) , the corresponding value of ω decreases from ω_0 , but the corresponding ellipsoids of revolution become more compressed. (d) For $M = M_1 < M_0$ or $\omega = \omega_1$ the ellipsoid of revolution coincides with the limiting ellipsoid of Jacobi. Now an ellipsoid having three different axes can be taken into account. These ellipsoids can be figures of equilibrium, when M varies between M_1 and ∞ , while the angular velocity decreases to 0, and the ratios of axes vary between certain limits.

Whether some figures of equilibrium can exist or not if ω surpasses the value ω_0 , was an important problem posed by Tchebysheff. Liapounov was able to prove in 1884, that there are no new figures of equilibrium in the neighborhood of the limit ellipsoid ($\omega = \omega_0$) but that there exist figures of equilibrium differing but little from some well-determined ellipsoids of Maclaurin and

Jacobi. As long as M is smaller than M_1 , the ellipsoids of revolution are stable. When M is larger than M_1 , Maclaurin's ellipsoids become unstable, but the ellipsoids of Jacobi remain stable. On investigating this property, Liapounov found that it holds only until a certain limit M_2 is reached. From the first approximation, the conclusion could be drawn that, if the value of M were increased, Jacobi's ellipsoids would change into figures of equilibrium of a new kind. These would be algebraic surfaces of the third order. One year later than Liapounov, Poincaré also discovered this new kind of figures of equilibrium.⁴ Liapounov expressed his doubt about the possibility of proving the existence of new figures of equilibrium by using the method of successive approximations, and only after twenty years did he make clear the reason for the difficulties met by the theory. In order to find a new figure of equilibrium which differs but little from an ellipsoid, one has to compare it, not to the given ellipsoid, but to a variable ellipsoid which is confocal to the former and passes through the point for which the value of potential is considered. This was the essential point of the new method developed by Liapounov. On using this method, the approximation of an arbitrary high order can be evaluated. Then, on applying the method of Cauchy, the convergence of successive approximations can be shown. Therefore, the existence of these figures of equilibrium can be proved with all the required precision. We shall later discuss in more detail the figures of a homogeneous fluid mass differing but little from the ellipsoids of Maclaurin and Jacobi.

Not connected with this group of investigations was the proof given by Sophie Kovalevsky concerning the existence of a ring-shaped figure of equilibrium. Despite the fact that a torus may be a figure of equilibrium of a homogeneous liquid mass, there are no applications of this result, since the figure is unstable. To explain the ring of Saturn, one will rather admit the Cassini hypothesis according to which this ring is composed of a swarm

⁴ C. R. Acad. Sci., Paris 1885, and Memoir in *Acta Mathematica*.

of solid particles. (See, for example, Lichtenstein [1], [2].)

The problem of figures of equilibrium becomes much more difficult if an inhomogeneous mass is considered. The first exact solution in a case where the density is a function of the pressure only was also given by Liapounov. It was published in the posthumous memoir of this investigator (1925–1927) and represents the final form of his method. Although it is a masterpiece of mathematical thought, it is, however, very difficult to study, since many details of calculations were omitted. As to the classical investigations, the first of them is represented by the memoir of Clairaut. The question it deals with is known as Clairaut's problem. To determine the compression of level surfaces and the distribution of gravity acceleration at the surface of a slow-rotating inhomogeneous fluid mass is its main purpose, a concentric stratification being assumed. In the same direction the problem was developed by Hamy, Véronnet, Wavre, and Dive, and certain particular cases were considered by Wiechert, Klussman, and Haalck.

The generalization of another kind was given by Liapounov [1], [2]. He assumes that a planet, say the Earth, is formed by a set of thin layers, each having a constant density and differing but little from a sphere. Since the density of these concentric layers might not be a continuous function of the radius of the layer, Liapounov considers an equation more general than that of Clairaut. To take into account the discontinuities of density, he made use for the first time, in this theory, of integrals representing a generalization of Riemann integrals in the sense that was determined also by Stieltjes. Liapounov [10] gave the following characteristics of methods used in the Clairaut problem. It has been known that in the second approximation, the level surfaces are determined as certain surfaces of revolution of the fourth order. Thus, it is not correct to look again for the elements of ellipsoids in the first approximation, since they are elements of those ellipsoids which themselves represent in the first approximation the unknown surfaces. This was the reason why Clairaut could not go further than the first approximation. The theories

of Legendre and Laplace were not subject to this objection, but Laplace made use of assumptions which can raise certain doubts. He assumes, namely, *a priori* that the unknown function can be developed in a series of spherical functions and makes use of such a series the convergence of which is questionable. If we put in the equation (1.3.3)

$$(1.4.11) \quad r = a(1 + \xi), \quad r' = a'(1 + \xi')$$

the potential U_M takes the form

$$(1.4.12) \quad U_M = \int_V \frac{\kappa' dV'}{D} = \int_0^A \kappa' a'^2 da' \int \frac{(1 + \xi')^2 \left(1 + \frac{\partial(a' \xi')}{\partial a'}\right) d\sigma'}{D}$$

Now, Laplace takes

$$(1.4.13) \quad \begin{aligned} \frac{1}{D} &= \frac{1}{r} \sum_0^\infty \left(\frac{r'}{r}\right)^n P_n(\cos \gamma') & \text{for } \frac{r'}{r} < 1 \\ \frac{1}{D} &= \frac{1}{r'} \sum_0^\infty \left(\frac{r}{r'}\right)^n P_n(\cos \gamma') & \text{for } \frac{r}{r'} < 1 \end{aligned}$$

where γ' is the angle between r and r' .

If $|\xi| < l$, the limits by means of which the convergence of these series is determined are

$$(1.4.14) \quad a' < a \frac{1-l}{1+l}, \quad a' > a \frac{1+l}{1-l}$$

Poisson pointed out that there is a layer

$$(1.4.15) \quad a \frac{1-l}{1+l} < a' < a \frac{1+l}{1-l}$$

for which the expressions (1.4.13) are not valid. This question was also investigated by Callandreau and later by Hopfner [1]. In connection with Liapounov's remarks the so-called desideratum of Tisserand [1] (Vol. II, p. 317) should also be mentioned. This desideratum was the starting point for investigations of Wavre, which will be discussed later.

In one particular case, namely, that of the Earth's figure, the theory has been presented in a form differing from those used

by other investigators. The problem posed by Bruns [1] is, namely, to investigate the level surfaces as those surfaces which are determined by general characteristics of the gravity potential and its derivatives. The deformations which actually occurred during the history of a planet are, of course, not taken into account, since Bruns considers the present distribution of masses in the Earth's interior and makes no assumptions about the connection between this distribution and the laws of Hydrostatics or some properties of the density. In this respect, the approach given by Bruns differs from that formulated by Clairaut. The latter identified the Earth's figure with one of figures of equilibrium. The problem of Bruns has been discussed in all details in the work of Hopfner [1].

An attempt has been made by Appell [2] to attack the problem of figures of equilibrium by a new method. He made use, namely, of the expression of potential in a point of the free surface given by Gauss. On writing

$$(1.4.16) \quad \int_s \frac{\alpha(\xi - x) + \beta(\eta - y) + \gamma(\zeta - z)}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}} d\sigma$$

and

$$(1.4.17) \quad \xi = f(\lambda, \mu), \quad \eta = \varphi(\lambda, \mu), \quad \zeta = \psi(\lambda, \mu)$$

it is

$$(1.4.18) \quad \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad \beta = \frac{B}{\sqrt{\dots}}, \quad \gamma = \frac{C}{\sqrt{\dots}},$$

$$A = \frac{D(\eta, \zeta)}{D(\lambda, \mu)}, \dots \quad d\sigma = \sqrt{A^2 + B^2 + C^2} d\lambda d\mu$$

Then, the equation of the free surface takes the form

$$(1.4.19) \quad \iint \begin{vmatrix} \xi-x & \eta-y & \zeta-x \\ \frac{\partial \xi}{\partial \lambda} & \frac{\partial \eta}{\partial \lambda} & \frac{\partial \zeta}{\partial \lambda} \\ \frac{\partial \xi}{\partial \mu} & \frac{\partial \eta}{\partial \mu} & \frac{\partial \zeta}{\partial \mu} \end{vmatrix} \frac{d\lambda d\mu}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} + \omega^2 s^2 - \text{const.}} = 0$$

On solving this integro-differential equation, one can determine the figure of equilibrium of a rotating homogeneous fluid mass. One unknown function will be defined by this equation if use is made of tangential coordinates. No further development of this method seems to be known.⁵

1.5. Conditions that Determine the Figures of Celestial Bodies

The question, of how to explain all those figures of celestial bodies which were observed, requires a further specification of conditions which might be essential for the formation of a body. If we eliminate from the list of celestial bodies all those having an irregular form like certain planetoids or nebulae, there will be different types of bodies the figures of which can be given a satisfactory explanation. In the short survey of methods used in the theory of figures of equilibrium given in the preceding section, we did not mention all conditions which must be taken into account. These conditions, however, will determine the degree of approximation to the actual state of a body, and, for example, in the case of the Earth, they may be very important for the understanding of the variation of its figure. The theory of figures of equilibrium is not able to yield an explanation of many characteristics in the Earth's structure which must be considered.

It is evident that the theoretically possible figures are determined by a set of conditions which can concern either the physical state of the mass of a celestial body, or its kinematical state, or form a group connected with the dynamical state in general. Because of basically different assumptions which have been made about the figures of celestial bodies, several groups of particular problems were obtained.

As to the physical state, the principal condition is that a celestial body is or was a fluid body. Most of those observed are at the present time in such a state and there are no objections in

⁵ More references concerning the development of methods will be given later.

general to the second part of our assumptions concerning the past periods. Certain exceptions, like the fact that the Saturn ring is composed of small solid particles, as it seems to be, will not diminish the importance of results based on the hypothesis of fluidity. It is evident also that the change in the physical state of a body may be followed by variations in its figure, as it may be asserted in case of the Earth. The transition to a plastic state or the solidification is a factor which must be taken into account. Despite the fact that changes in figures produced by these processes are normally small as compared with the size of a celestial body, they are very important in case of the Earth.

The second group of conditions refers to the kinematical state. In the theory of figures of equilibrium, we had a simple example of a postulated state of motion or rest. If a given fluid mass rotates about an axis having a direction fixed in space with a constant angular velocity, its figure does not vary. The constant angular velocity is, however, not a necessary condition for the invariance of the figure. Liquid bodies not changing their shape can exist if there are internal movements which follow, for example, a distribution of velocities given by a law, $\omega = F(s^2, z)$, that is, which have a zonal rotation. There may also be other important kinematical conditions concerning some regular kind of deformation. Postulated pulsations, oscillations of some other type, or progressing variations belong to this group of conditions.

In the third group of conditions, the law of gravitation must be mentioned in the first place along with the influence of other bodies. We may assume, obviously, any law of gravitation, but in all classical investigations the use has been made of Newton's law and only this law is here considered. If the influence of other celestial bodies is neglected, we have the simple case of an isolated fluid mass. Conditions of equilibrium of a system of two or more fluid bodies have been, of course, also considered, and in more general problems, we have to take into account, besides the rotation of each body, its revolution and the variations of both of them like those taking place in the solar system. Certain

approximations in problems of this kind were possible because of very large distances between the members of a system.

Taking into account the great variety of conditions, a certain classification of problems may present an interest and is now given. When we consider a figure of a celestial body, its free surface is not, in general, the only object of an investigation. The stratification is sometimes also of importance and, therefore, the term figure will from now on be used in the more general sense which includes the stratification. The problems which were more or less investigated are as follows:

A. An isolated mass.

I. The figures of equilibrium:

1. of a homogeneous fluid mass,
2. of a heterogeneous fluid mass,
3. of a system having a solid (rigid) core and a fluid envelope,
4. of a fluid mass with floating solids,
5. of a fluid in a solid crust,
6. of a plastic mass,
7. of an elastic solid.

II. The invariable figures:

1. of a zonal rotating homogeneous fluid mass,
- 2-7. of a zonal rotating mass corresponding to conditions given in I., 2-7,
8. of a mass under some other conditions.

III. The varying figures:

1. of a homogeneous fluid mass,
2. of each system mentioned above.

B. A system of separated bodies:

1. a fluid mass and a mass center,
2. system Earth and Moon,
3. double-star system,
4. more complicated systems.

Of course, we are able to present only a short selection of problems. An attempt is however made to cover the results from the whole field.

The most simple problem on this list is the problem of figures of equilibrium which an isolated homogeneous liquid mass can have. As mentioned previously, it presented great difficulties, and it is not solved yet in all details. The methods of solution applied in this problem as well as in the case of an inhomogeneous fluid reached such a degree of perfection that certain of them belong to the most advanced mathematical tools. However there are but few of the problems listed above which are attacked by similar methods. It is hoped that more exact mathematical methods will be applied in all branches of the theory.

Two directions in which the problem of equilibrium of a fluid mass is developed at the present time should be mentioned here. Most of the investigations are dealing with the equilibrium and small deformations of gaseous masses. Since stars are such masses, their stratification is the main object of the study. Some general conclusions reached about the stratification were also applied to planets. However, for these celestial bodies and especially for the Earth, more attention must be paid to factors like increasing viscosity or plasticity and to different periods in the solidification.

1.6. Integral Equation of Liouville

Since in some of the investigations which will be discussed, use has been made of the so-called Liouville formulas (Liouville [1], [2]), they will be derived in this section. These formulas represent solutions of an equation which belongs to the class of homogeneous integral equations. The general form of this equation is

$$(1.6.1) \quad \int \frac{l' \zeta' d\Omega'}{D} = m\zeta$$

where the integral is taken over a certain given surface. The element of this surface is $d\Omega'$ and D is the distance between two

points, one of them (M') being a point of the surface. As usual ζ is an unknown function of coordinates of the other point (M) ζ' its value at M' . The factors l' and m will be specified immediately. On considering a unit sphere Σ and its element $d\sigma'$ we can put $d\sigma' = l' d\Omega'$ and integrate in (1.6.1) over the unit sphere. The particular case important for the theory is that of an ellipsoid, and for this case we will use the notation

$$(1.6.2) \quad \chi = \nu \int \frac{\chi' d\sigma'}{D} = \nu \int \frac{l' \chi' d\sigma'}{D}$$

where we have to integrate over the corresponding unit sphere or over the surface of this ellipsoid respectively.

It has been shown by Liouville that the solutions of the equation (1.6.2) can be given in terms of Lamé functions. Since in other sections we shall make use of special notations, we introduce some of them in the formulae which follow. The distance $D = MM'$, in the expression (1.6.1) or (1.6.2) interpreted as a potential will be written in the form $D(a, 1)$. The new parameter a is one of those which will be used in order to determine the position of a point. The value $a' = 1$ refers to those points with respect to which we integrate. The corresponding complete expressions will be given later. Now, let R, S, M , and N be the Lamé functions⁶ of the n th order and R' and S' the values of the first and second functions corresponding to $a' = 1$. Then, the solutions of the equation

$$(1.6.3) \quad \chi = \nu \int \frac{\chi' d\sigma'}{D(a, 1)}$$

are given by the formulae:

⁶ The inconvenience that the Lamé function M has the notation coinciding with that of the point M is increased by the fact that the momentum in Section 1.4 is denoted by the same letter. It is, however, very difficult to change many standard notations to achieve a unification. Since in every method there are many concepts involved, it was possible to make only few adjustments. The hope can be expressed only that the use of the same letter to denote different quantities in different chapters will not produce much confusion.

$$\begin{aligned}
 (1.6.4) \quad & \int \frac{l' M' N' d\Omega'}{D(a, 1)} = \frac{4\pi}{2n+1} RSMN \quad \text{if } a=1, M \text{ is on the ellipsoid} \\
 & \int \frac{l' M' N' d\Omega'}{D(a, 1)} = \frac{4\pi}{2n+1} RS'MN \quad \text{if } a<1, M \text{ is an internal point} \\
 & \int \frac{l' M' N' d\Omega'}{D(a, 1)} = \frac{4\pi}{2n+1} R'SMN \quad \text{if } a>1, M \text{ is an external point}
 \end{aligned}$$

The first member in every equation (1.6.4) represents the value of the potential of an ellipsoid at a point $M(a, \vartheta, \psi)$, if the density of this surface is equal to $l' M' N'$, ϑ and ψ being spherical angles. From the integral equation (1.6.3) and its solution (1.6.4) it follows that the products MN are the eigenfunctions and the expressions

$$(1.6.5) \quad \nu_{n,s} = \frac{2n+1}{4\pi} \frac{1}{RS}$$

are the eigenvalues.

For an ellipsoid of revolution, we have

$$(1.6.6) \quad \int \frac{M' N' d\sigma'}{D(a, 1)} = \frac{4\pi}{2n+1} \bar{R}SMN$$

or

$$(1.6.6') \quad \int \frac{Y'_{n,s} d\sigma'}{D(a, 1)} = \frac{4\pi}{2n+1} \bar{R}SY_{n,s}$$

where $Y_{n,s}$ are spherical harmonics. If we put $\bar{\mu} = \cos \vartheta$, these functions are expressed in terms of associated Legendre functions $P_{n,k}$ which themselves depend on the Legendre polynomials P_n . By definition, it is

$$(1.6.7) \quad P_n(\bar{\mu}) = \frac{1}{2 \cdot 4 \dots 2n} \frac{d^n(\bar{\mu}^2 - 1)^n}{d\bar{\mu}^n}$$

and

$$(1.6.8) \quad P_{n,k}(\bar{\mu}) = (\sqrt{1 - \bar{\mu}^2})^k \frac{d^k P_n(\bar{\mu})}{d\bar{\mu}^k}$$

Then, the spherical harmonics of the n th order are determined

as follows:

$$(1.6.9) \quad \begin{array}{l} Y_{n,2k}: P_n(\bar{\mu}), P_{n,1}(\bar{\mu}) \cos \psi, P_{n,2}(\bar{\mu}) \cos 2\psi, \dots, P_{n,n}(\bar{\mu}) \cos n\psi \\ Y_{n,2k-1}: P_{n,1}(\bar{\mu}) \sin \psi, P_{n,2}(\bar{\mu}) \sin 2\psi, \dots, P_{n,n}(\bar{\mu}) \sin n\psi \end{array}$$

The eigenvalues in (1.6.6') can also be given in terms of Legendre's functions. It is

$$(1.6.10) \quad \bar{R}S = \frac{\sqrt{\varrho}}{\gamma_{n,s}} \int \frac{[P_{n,s}(\bar{\mu}) \cos s\psi]^2}{\varrho + \bar{\mu}^2} d\sigma$$

where

$$(1.6.11) \quad \gamma_{n,s} = \int [P_{n,s}(\bar{\mu}) \cos s\psi]^2 d\sigma$$

and the variable ϱ will be determined later. A detailed theory of Lamé functions and spherical harmonics can be found, for example, in Vol. IV of Appell [1].

CHAPTER II

Figures of Equilibrium; Inverse Method

2.1. The Fundamental Equation

A brief account of classical investigations concerning the figures of celestial bodies will be given in this chapter. If we assume that a celestial body is changing its shape, the problem of its motion becomes, in general, too intricate, since in the equation of motion the free surface of the body involved is unknown, and in the case of an inhomogeneous body, even the whole stratification is unknown. Thus, the motion will be produced by variable forces depending on a varying stratification. Even in the simplest case representing the equilibrium of a liquid mass, by which the figures of many celestial bodies can be explained, the problem has been reduced to the solution of a functional equation. New, very complicated, methods were required to solve this type of equation, and one can easily see, therefore, why the first solutions were found by the inverse method. Attempts were made, namely, to investigate whether the most simple geometric figures can be free surfaces of a fluid mass or not. It had to be proved only that all the conditions of the problem will be satisfied by such an assumption. For example, a figure of equilibrium of a fluid mass at rest is determined by equation (1.1.9).

$$(2.1.1) \quad \int_V \frac{\kappa' dV'}{D} = \text{const.}$$

In this functional equation, the limits of V are unknown and so is the density, if the fluid is inhomogeneous. It may be easily seen, however, that, on assuming that the figure of a homo-

geneous mass is a sphere or that the stratification is represented by a set of concentric spheres such that at the surface of each of them the density has a constant value, the conditions of equilibrium can be satisfied.

For the figures of equilibrium of a rotating fluid mass we have the equation (1.1.8)

$$(2.1.2) \quad \int_V \frac{\kappa' dV'}{D} + \frac{\omega^2 s^2}{2f} = \text{const.}$$

or for a homogeneous mass

$$(2.1.3) \quad \int_V \frac{dV'}{D} + \frac{\omega^2 s^2}{2f\kappa} = \text{const.}$$

Now, we can see that no sphere satisfies these conditions. But, in case of a homogeneous fluid mass, ellipsoids could be among the possible figures of equilibrium. Of course, using the inverse methods, we can hardly see whether all solutions are found or not. As mentioned previously, the exact proofs of the fact that a sphere is a unique figure of equilibrium of a nonrotating fluid were given by Liapounov and Carleman. For a rotating fluid, however, we know many solutions. They represent ellipsoids, ring-shaped figures, other figures differing but little from the ellipsoids, and it is not yet shown that this list is complete.

2.2. Maclaurin's and Jacobi's Ellipsoids

The basic equations of the theory have been mentioned in Section (1.4). We now write the equations (1.4.1)–(1.4.4) of that section. If the expression for the potential of a homogeneous ellipsoid at a point $M(x, y, z)$

$$(2.2.1) \quad fU = \text{const} - L_x \frac{x^2}{2} - L_y \frac{y^2}{2} - L_z \frac{z^2}{2}$$

is inserted into the equation (1.1.8) the conditions for the surface of equal pressure are obtained in the form

$$(2.2.2) \quad (L_x - \omega^2)x^2 + (L_y - \omega^2)y^2 + L_z z^2 = C$$

Furthermore, if the surface of an ellipsoid of revolution

$$(2.2.3) \quad \frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$$

is identified with the free surface of the fluid

$$(2.2.2') \quad (L_x - \omega^2)(x^2 + y^2) + L_z z^2 = C$$

the conditions which must be satisfied are: $L_x = L_y$ and

$$(2.2.4) \quad a^2(L_x - \omega^2) = c^2 L_z = C$$

On deriving further conditions, it is convenient to introduce a new variable l determined by the equation

$$(2.2.5) \quad a^2 = c^2(1 + l^2)$$

Then, by (2.2.4) and (2.2.5), we obtain

$$(2.2.6) \quad (1 + l^2)(L_x - \omega^2) = L_z$$

For an ellipsoid of revolution having the axes $2a$ and $2c$ and a constant density κ the coefficients $L_x = L_y = L$ and L_z in the expression of the potential (1.4.1) are given by the formulae

$$(2.2.7) \quad \begin{aligned} L &= 2\pi f \kappa a^2 c \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)^2 (c^2 + \lambda)^{3/2}} \\ L_z &= 2\pi f \kappa a^2 c \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(c^2 + \lambda)^{3/2}} \end{aligned}$$

(see, for example, Appell [1], Vol. IV, p. 48). If a new variable (t) is introduced and $\lambda = c^2 t$, L and L_z can be expressed in terms of elementary functions. It is

$$(2.2.8) \quad \begin{aligned} L &= 2\pi f \kappa (1 + l^2) \int_0^\infty \frac{dt}{(1 + l^2 + t)^2 \sqrt{1 + t}} \\ &= 2\pi f \kappa \frac{1 + l^2}{l^3} \left(\tan^{-1} l - \frac{l}{1 + l^2} \right) \\ L_z &= 4\pi f \kappa \frac{1 + l^2}{l^3} (l - \tan^{-1} l) \end{aligned}$$

The mass of the ellipsoid is also a function of the variable l

$$(2.2.9) \quad m = \frac{4}{3} \pi \kappa a^2 c = \frac{4}{3} \pi \kappa c^3 (1 + l^2) = \frac{4\pi \kappa a^3}{3\sqrt{1 + l^2}}$$

Now, if the values of m , κ , and l are given, we can find from this equation a and from (2.2.5) the other axis of the ellipsoid. On writing (2.2.4) in the form

$$a^2 L - c^2 L_z = a^2 \omega^2 > 0$$

and inserting (2.2.7), we obtain

$$(a^2 - c^2) \int_0^\infty \frac{\lambda d\lambda}{(a^2 + \lambda)^2 (c^2 + \lambda) \sqrt{c^2 + \lambda}} > 0$$

It is obvious that the integral is positive and, therefore, $a > c$. Thus, if an ellipsoid is a figure of equilibrium, it must be an oblate one.

On eliminating now L and L_z from (2.2.4) we obtain the condition

$$(2.2.10) \quad \frac{\omega^2}{2\pi f \kappa} = \frac{(3 + l^2) \tan^{-1} l - 3l}{l^3} \equiv h(l)$$

which is to be satisfied by ω . If a value of l is given, the corresponding ω can be found by (2.2.10). From the last equation (2.2.4) the constant C , which will determine the value of potential at the free surface, also becomes known. The relationship (2.2.10) may be represented graphically. Substituting $\tan^{-1} l = l - l^3/3 + \dots$ it is easy to see that the curve $h = h(l)$ which begins at the point O , will reach a maximum and approach asymptotically to the l axis. Some other conclusions concerning ω were mentioned in Chapter I.

In case of ellipsoids of Jacobi, assuming that $c < b < a$, the coefficients L_x , L_y , and L_z are given by the formulae (Appell [1], p. 54).

$$(2.2.11) \quad \begin{aligned} L_x &= 2\pi f \kappa a b c \int_0^\infty \frac{d\lambda}{(a^2 + \lambda) G(\lambda)}, \quad L_y = 2\pi f \kappa a b c \int_0^\infty \frac{d\lambda}{(b^2 + \lambda) G(\lambda)}, \\ L_z &= 2\pi f \kappa a b c \int_0^\infty \frac{d\lambda}{(c^2 + \lambda) G(\lambda)}, \quad G^2(\lambda) = (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \end{aligned}$$

Instead of (2.2.3) and (2.2.4) we now have

$$(2.2.12) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and, by (2.2.2) and (2.2.12)

$$(2.2.13) \quad a^2(L_x - \omega^2) = b^2(L_y - \omega^2) = c^2 L_z = C$$

Hence

$$(2.2.14) \quad \omega^2 = \frac{a^2 L_x - b^2 L_y}{a^2 - b^2}$$

and

$$(2.2.15) \quad a^2 b^2 (L_x - L_y) + c^2 (a^2 - b^2) L_z = 0$$

Inserting (2.2.11) in (2.2.14), we obtain

$$(2.2.16) \quad \frac{\omega^2}{2\pi f \kappa} = abc \int_0^\infty \frac{\lambda d\lambda}{(a^2 + \lambda)(b^2 + \lambda)G(\lambda)}$$

and by (2.2.15)

$$(2.2.17) \quad (b^2 - a^2) \int_0^\infty \left[\frac{a^2 b^2}{(a^2 + \lambda)(b^2 + \lambda)} - \frac{c^2}{c^2 + \lambda} \right] \frac{d\lambda}{G(\lambda)} = 0$$

This condition is identically satisfied by $b = a$. Assuming that $b \neq a$ the substitution

$$\frac{c^2}{a^2} = s, \quad \frac{c^2}{b^2} = t, \quad \lambda = c^2 \tau$$

yields the expressions

$$(2.2.18) \quad \frac{\omega^2}{2\pi f \kappa} = st \int_0^\infty \frac{\tau d\tau}{(1 + s\tau)(1 + t\tau)\Delta} = h(s, t),$$

$$\Delta^2 = (1 + s\tau)(1 + t\tau)(1 + \tau)$$

and

$$(2.2.19) \quad 0 = (1 - s - t) \int_0^\infty \frac{\tau d\tau}{\Delta^3} - st \int_0^\infty \frac{\tau^2 d\tau}{\Delta^3} = g(s, t)$$

The variables s and t have positive values according to our definition, and the integrals in the last equation are also positive.

By (2.2.19) if the given ellipsoid is a figure of equilibrium, it must be that $s + t < 1$.

The mass of the ellipsoid can also be expressed in terms of parameters s and t . We have, namely

$$(2.2.20) \quad m = \frac{4}{3} \pi \kappa abc = \frac{4}{3} \pi \kappa \frac{c^3}{\sqrt{st}} = \frac{4}{3} \pi \kappa \frac{b^3 t}{\sqrt{s}} = \frac{4}{3} \pi \kappa \frac{a^3 s}{\sqrt{t}}$$

If the mass, the density, and the values of parameters s and t are given, we can compute all axes and the corresponding angular velocity, but we have always to take into account that by (2.2.19), s is a function of t . Equations (2.2.18) and (2.2.19) represent a curve in the space of variables s, t, h , and the equation $g(s, t) = 0$ gives its projection on the plane st . The complete discussion of these conditions can be found in Vol. II of Tisserand [1] or Appell [1], Vol. IV, pp. 61–69. The function $h = h(s, t)$ reaches a maximum h_0 at $s = t = t_0 = 0.3396$ and the corresponding ellipsoid belongs to the series of Maclaurin's ellipsoids. This maximum $h_0 = 0.1871$ is the limit value (1.4.9) which determined the point of intersection of two curves representing sets of the ellipsoids of Jacobi and those of Maclaurin.

2.3. Inhomogeneous Figures

The first investigations of Clairaut concerning the figures of equilibrium of an inhomogeneous mass were mentioned in section (1.4). His well-known equation representing the connection between the average density of a layer and its ellipticity as well as some related theorems were for a long time the single base of the theory of inhomogeneous figures of equilibrium. To obtain these results, we begin, however, with more general assumptions concerning the stratification in a mass. We follow the derivation given by Wavre [1].

If we write the equations (1.1.3) in the form

$$(2.3.1) \quad \frac{1}{\kappa} \frac{\partial \phi}{\partial x} = X + \omega^2 x, \quad \frac{1}{\kappa} \frac{\partial \phi}{\partial y} = Y + \omega^2 y, \quad \frac{1}{\kappa} \frac{\partial \phi}{\partial z} = Z$$

they show that gravity force, i.e., the resultant of attraction and

of the centrifugal force has the direction of the normal to the surface of equal pressure p (x, y, z) = const. In case $\kappa = \varphi(p)$, three sets of surfaces: $p = \text{const.}$, $\kappa = \text{const.}$, and the level surfaces coincide as it was shown earlier.

We shall now consider this case and denote the gravity potential by

$$(2.3.2) \quad W = \int \frac{dp}{\kappa} + \text{const.} = W(\kappa)$$

A stratification is usually taken which corresponds to a density distribution such that κ can be expressed in terms of a parameter which we call \hat{a} . Its sense will be specified later.

Then, the gravity potential, as well as the gravity g become functions of this parameter $W(\hat{a})$ and $g(\hat{a})$, respectively. The coordinates of the vector \mathbf{g} are obviously given by the second members in (2.3.1) or by the equations

$$(2.3.3) \quad g_x = g\alpha = \frac{\partial W}{\partial x}, \quad g_y = g\beta = \frac{\partial W}{\partial y}, \quad g_z = g\gamma = \frac{\partial W}{\partial z}$$

α, β, γ being the angular coefficients of the interior normal of the surface $W = \text{const.}$ If dn is the element of this normal, we have

$$(2.3.4) \quad \frac{dg}{dn} = \frac{\partial g}{\partial x} \frac{dx}{dn} + \frac{\partial g}{\partial y} \frac{dy}{dn} + \frac{\partial g}{\partial z} \frac{dz}{dn} = \mathbf{n} \cdot \nabla g$$

and, by differentiation of (2.3.3), we easily obtain

$$(2.3.5) \quad \nabla^2 W = \frac{dg}{dn} + g \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right)$$

The expression in brackets represents the $\text{div } \mathbf{n}$ if \mathbf{n} is a unity vector in the direction of the normal. Now, it is

$$(2.3.6) \quad \text{div } \mathbf{n} = -K$$

where the value of K is the double average curvature of the surface $W = \text{const.}$ Thus

$$(2.3.7) \quad \nabla^2 W = \frac{dg}{dn} - Kg$$

and by (1.1.5), it is

$$\nabla^2 W = \nabla^2 fU + \nabla^2 \frac{\omega^2 s^2}{2}$$

Since the potential fU satisfies the Poisson equation, we obtain the equation of Bruns [1]

$$(2.3.8) \quad \frac{dg}{dn} - Kg = -4\pi f\kappa + 2\omega^2$$

This is an exact equation which yields a value of one of the quantities involved if the others are known. It may be easily seen from the equations (2.3.3)–(2.3.8) that the relationship (2.3.8) does not depend on any hypothesis about the connection between three sets of surfaces mentioned above. Some other conclusions can be drawn from (2.3.8). Suppose that dn and $d\hat{a}$ are both positive at the same time, i.e., we change now the sense of dn which was positive inwards and put $dn/d\hat{a} = N$. Then it is

$$(2.3.9) \quad -gN = W' = \frac{dW}{d\hat{a}}$$

g being the derivative of W along the normal. Along a surface

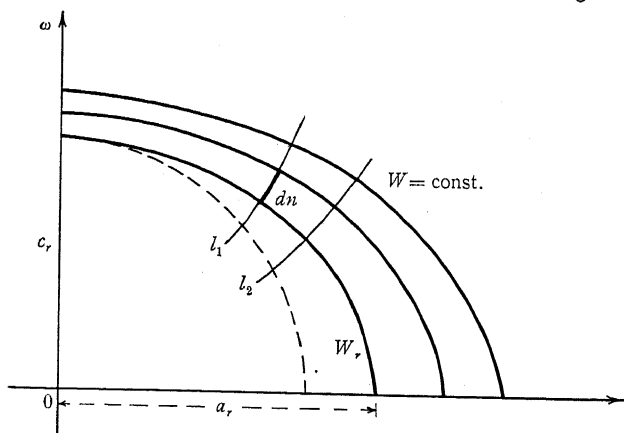


Figure 1

$W = \text{const.}$ the expression (2.3.9) does not change and therefore

we can write

$$(2.3.10) \quad (gN)_1 = (gN)_2$$

This holds obviously when the corresponding points M_1 and M_2 are displaced along the lines of force l_1 and l_2 respectively (Figure 1). By differentiating (2.3.10) we obtain

$$(2.3.11) \quad \left(\frac{dg}{d\hat{a}} N + g \frac{dN}{d\hat{a}} \right)_1 = \left(\frac{dg}{d\hat{a}} N + g \frac{dN}{d\hat{a}} \right)_2$$

and from (2.3.8) and (2.3.9)

$$(2.3.12) \quad \frac{dg}{d\hat{a}} = KW' + (-4\pi f\kappa + 2\omega^2)N$$

On inserting (2.3.12) into (2.3.11) it is

$$(2.3.13) \quad (4\pi f\kappa - 2\omega^2)(N_2^2 - N_1^2) = W' \left[\left(KN + \frac{1}{N} \frac{dN}{d\hat{a}} \right)_1 - \left(KN + \frac{1}{N} \frac{dN}{d\hat{a}} \right)_2 \right]$$

That this equation can be subject to further useful transformations has been shown by Wavre, but we restrict ourselves by mentioning this formula only. As to the shape of the surfaces of equal density, we did not make any assumption as yet. It has been proved by Volterra [2] for an infinite number of layers and for a finite set of such layers by Hamy in 1887 that these surfaces cannot coincide with a set of homothetic ellipsoids. Volterra's statement holds when the density is an integrable function. Nevertheless, ellipsoids are used as surfaces at least approximately representing stratification. Some conclusions drawn from this assumption are discussed in the next section.

2.4. Clairaut's Equation and Some Theorems

We shall now assume that (a) $W = \text{const.}$ are closed surfaces each enclosing all preceding in the set, (b) they have a common center, and (c) that they are surfaces of revolution. The equation (2.3.13) can be applied to a surface $W_r = \text{const.}$ The polar and equatorial "radius" of this surface are $c_r(\hat{a})$, $a_r(\hat{a})$ respectively.

The average curvature at its pole is $K_P(\hat{a})$ and at the equator $K_E(\hat{a})$. Let $g_P(\hat{a})$ be the gravity at the pole P_r . Then we can write (2.3.13) in the form

$$(2.4.1) \quad (4\pi f \kappa - 2\omega^2) \left[\left(\frac{da_r}{d\hat{a}} \right)^2 - \left(\frac{dc_r}{d\hat{a}} \right)^2 \right] \\ = -g_P \frac{dc_r}{d\hat{a}} \left[K_P \frac{dc_r}{d\hat{a}} + \left(\frac{d\hat{a}}{dc_r} \right) \frac{d^2 c_r}{d\hat{a}^2} - K_E \frac{da_r}{d\hat{a}} - \left(\frac{d\hat{a}}{da_r} \right) \frac{d^2 a_r}{d\hat{a}^2} \right]$$

According to the assumption of Clairaut, the stratification is determined by the set of concentric ellipsoids, and the parameter \hat{a} represents their polar semi-axes. Then, the curvatures are approximately

$$K_E = \frac{2}{\hat{a}}, \quad K_P = \frac{2}{\hat{a}} \left(1 - 2 \frac{\delta}{\hat{a}} \right)$$

where $\delta = a_r - c_r$ is small for ellipsoids differing but little from a sphere. By neglecting the terms of the order of δ^2 , equation (2.4.1) takes the form

$$(2.4.2) \quad \frac{4\pi f \kappa - 2\omega^2}{g_P} \delta' = \frac{\delta'}{\hat{a}} + \frac{2}{\hat{a}^2} \delta - \frac{\delta''}{2}$$

Furthermore, if ω^2 is small, δ , δ' , and δ'' are also small, the product $\omega^2 \delta'$ can be neglected and the average density κ_a in the interior of $W_r = \text{const.}$ as well as the gravity can be computed approximately by assuming a spherical distribution of masses. Thus we will have

$$m_r = 4\pi \int_0^r \kappa r^2 dr = \frac{4\pi}{3} r^3 \kappa_r \quad r = c_r$$

It follows from this equation that

$$3c_r^2 \kappa_r + c_r^3 \kappa'_r = 3c_r^2 \kappa$$

and

$$(2.4.3) \quad \frac{3}{c_r} + \frac{\kappa'_r}{\kappa_r} = \frac{4\pi f \kappa}{g_P}$$

since the denominator in the second member $\frac{4}{3}\pi/c_r \kappa_r$ represents

the gravity at the surface of the sphere. Thus, since $c_r = \hat{a}$, we obtain by (2.4.2) and (2.4.3)

$$(2.4.4) \quad \frac{2}{\hat{a}} + \frac{\kappa'_r}{\kappa} = \frac{2}{\hat{a}^2} \frac{\delta}{\delta'} - \frac{1}{2} \frac{\delta''}{\delta'}$$

The ellipticity $e = \delta/\hat{a}$ can be introduced as a new variable. Then the Clairaut equation takes the form

$$(2.4.5) \quad e'' \kappa_r + \frac{6}{\hat{a}} e' \kappa_r + 2e' \kappa'_r + \frac{2}{\hat{a}} e \kappa'_r = 0$$

It is also known that for the gravity at a pole Clairaut's theorem can be proved. It is expressed by the conditions

$$(2.4.6) \quad \frac{4}{5} e \leq \frac{\omega^2 \hat{a}}{g} \leq 2e, \quad \frac{1}{2} \frac{\omega^2 \hat{a}}{g} \leq e \leq \frac{5}{4} \frac{\omega^2 \hat{a}}{g}$$

Two further relationships can be also proved, the second is very important in applications. It is, namely,

$$(2.4.7) \quad e + \frac{g_{\text{equa}} - g_{\text{pole}}}{g} = \frac{5}{2} \frac{\omega^2 \hat{a}}{g}$$

where g is the gravity at some latitude on the surface of a planet.

All these results are obtained for those stratifications which were assumed in this section. We shall also mention here another classical relationship known as the equation of D'Alembert

$$(2.4.8) \quad C - A = \frac{2}{3} \frac{g}{f} \hat{a}^4 \left(e - \frac{1}{2} \frac{\omega^2 \hat{a}}{g} \right)$$

where A and C are moments of inertia of a planet with respect to equatorial and polar diameter respectively.

As mentioned above, when the problem of a figure of an inhomogeneous planet is reduced to the form given by Clairaut, it is assumed that the level surfaces are concentric ellipsoids and that the distribution of densities is known. The problem is then to determine the variation of ellipticity from the free surface toward the center. We mentioned a quite different approach to the problem of the figure of the Earth made by Bruns [1]. The

basic definitions and assumptions of his theory are as follows. The level surfaces represent a set of surfaces perpendicular to the force of gravity at each point of the Earth. Because of rotation these surfaces are determined by the function $W = fU + Q$, i.e., by the equation $W(x, y, z) = \text{const.}$ Then the gravity satisfies the equation (2.3.8). We note first that under certain assumptions concerning the mass of the Earth which seem to be close to the actual conditions, Pizzetti [1] has proved that the level surfaces form a linear set of closed surfaces each enclosing the preceding ones. This fact had been considered at the beginning of this section as an assumption. The further investigation of level surfaces by the methods of potential theory has been brought in connection with geodetic data (see, for example, Hopfner [1]).

We must restrict ourselves by quoting only few conclusions of Bruns. No level surface can be well approximated by a single analytical surface having a simple shape. As to the figure of the Earth, a level spheroid

$$\hat{W} = \text{const.} = W_0 = \frac{Y_0}{r} + \frac{Y_2}{r^3} + \frac{\omega^2 s^2}{2f}$$

should be first considered. It differs but little from an ellipsoid of revolution. Next approximation is then the geoid $W = \hat{W} + W_1 = W_0$. Hence at a point of the geoid the term \hat{W} should not be equal to W_0 . Now the expression W_1 is the sum of all terms of higher orders in the expansion of the attraction potential in a series of spherical harmonics. If ζ is the distance between these two figures which has been measured along the geoid's normal it can be proved that it is determined by the equation

$$\zeta = - \frac{W_1}{\bar{g} \cos \varepsilon}$$

where \bar{g} is the gravity at the level spheroid and ε the angle between the two normals. This is Bruns' theorem. To find the distortion represented by the geoid from the apparent differences in gravity $g - \bar{g}$ is the well-known problem of Stokes.

2.5. *Dirichlet's Ellipsoid; Investigations of Riemann et al.*

As the last example of the classical discussion in the theory of figures of celestial bodies, we shall take the ellipsoid of Dirichlet. Assuming that at an initial instant a homogeneous liquid mass has a shape of an ellipsoid

$$(2.5.1) \quad \frac{\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} + \frac{\hat{z}^2}{c^2} = 1$$

Dirichlet considered the motion determined by the condition that the coordinates of an element x, y, z are linear functions of $\hat{x}, \hat{y}, \hat{z}$.

$$(2.5.2) \quad x = l\hat{x} + m\hat{y} + n\hat{z}, \quad y = \dots, \quad z = \dots$$

where l, m, n are functions of time. As usual the particles are subjected to the mutual attraction. The pressure at the surface of the liquid can be in general a function of time $P(t)$. Inserting expressions (2.5.2) in equation (2.5.1) we see that the outer surface of liquid is a variable ellipsoid concentric to (2.5.1) and the equations of motion are satisfied if the pressure

$$(2.5.3) \quad p = P(t) + \sigma(t) \left[1 - \frac{\hat{x}^2}{a^2} - \frac{\hat{y}^2}{b^2} - \frac{\hat{z}^2}{c^2} \right]$$

This motion of the liquid can be split into two parts: first, in a rigid rotation with respect to an axis passing through the center and second, the motion of particles relative to the rotating axes of reference $O\xi\eta\zeta$. The relative velocities have to be proportional to the new coordinates, ξ, η, ζ

$$(2.5.4) \quad v_\xi = \lambda_1 \xi, \quad v_\eta = \lambda_2 \eta, \quad v_\zeta = \lambda_3 \zeta$$

For the particular case where there is no rotation, Dirichlet obtained isochronous oscillations in which the shape of the liquid is varying from an elongated to an oblate ellipsoid on taking at some intermediate instant the spherical shape. If the initial velocity is not equal to zero and does not satisfy some special condition, the corresponding ellipsoid becomes indefinitely flattened or stretched.

If the initial velocity is below a certain limit, the angular velocity of the rotating liquid can vary between two limits. However, for a larger value of the angular velocity of an elongated ellipsoid a sufficiently great external pressure must be applied. The existence of ellipsoids of Maclaurin and Jacobi has been proved by Dirichlet as a special case of his theory. In an additional note, Dedekind derived from the investigation of Dirichlet a very interesting result. Each ellipsoid of Jacobi, namely, will preserve its shape and *position* if the internal movements of particles are given by the equations

$$(2.5.5) \quad x = \hat{x} \cos \omega t + \hat{y} \frac{a}{b} \sin \omega t, \quad y = -\hat{x} \frac{b}{a} \sin \omega t + \hat{y} \cos \omega t, \quad z = \hat{z}$$

The constant ω is then the angular velocity of the corresponding ellipsoid of Jacobi (2.1.18). Each particle, therefore, describes an ellipse, the equation of which is given in the parametric form (2.5.5).

Riemann [1] improved the unfinished investigation of Dirichlet and modified the problem. He studied the characteristic changes of the axes of the ellipsoid and the relative motion of the liquid with respect to them by eliminating the initial moment from the equations of motion. Then, it could be proved that there are only four particular cases known from previous investigations in which the principal axes of ellipsoids do not change. As to the small oscillations, there are two types of changes which differ in stability. According to Riemann, if the disturbed surface of Maclaurin's ellipsoid remains an ellipsoid of revolution, the equilibrium is stable.

In case the circular equator becomes an ellipse and the polar axis remains unchanged, there is instability if the eccentricity exceeds certain limits.

The problem of a varying homogeneous liquid ellipsoid was further investigated by Brioschi, Lipschitz, Greenhill, Basset, Tedone, Love, Stekloff, and Hargreaves. (See, for example, H. Lamb *Hydrodynamics*, p. 722).

CHAPTER III

Method of Poincaré

3.1. Fundamental Functions

Poincaré in his famous memoir [1] suggested a new method to solve the problem of figures of equilibrium of a rotating fluid mass. This method yields solutions representing also figures of a new kind which are usually called the “figures of Poincaré.” As mentioned previously, in the series of Maclaurin’s ellipsoids there is one corresponding to the so-called branch point from which the series of Jacobi’s ellipsoids start. The question arises whether there are other branch points of a similar kind. The answer given by Poincaré was positive. It is necessary to note that Liapounov discovered the same figures of equilibrium one year earlier than Poincaré. He did not decide to insist on their existence in his first paper. Nevertheless it seems more appropriate to call these figures “Liapounov-Poincaré’s figures of equilibrium,” on taking also into account that the former investigator later applied most exact methods to solve several problems concerning these and other figures of equilibrium.

In this chapter we shall discuss the method of Poincaré. An important part in it is played by the “fundamental functions” associated with a certain closed surface S . They are known now as eigenfunctions, but the terminology of the theory of integral equations was introduced in the investigations of Lichtenstein which will be discussed later.

To see the conditions which fundamental functions must satisfy we have to make use of well-known properties of a surface potential due to the Newton’s gravitation or to Coulomb’s electric forces.

This potential, namely, is continuous on the surface S , but its normal derivative is discontinuous.

Now, we will determine a set of functions U_k which have the properties as follows: (a) The value $U_k^{(i)}$ in the space τ_i bounded by S is a harmonic function

$$(3.1.1) \quad \Delta U_k^{(i)} = 0$$

(b) The value $U_k^{(e)}$ of the function U_k in space τ_e exterior to S satisfies also the Laplace equation

$$(3.1.2) \quad \Delta U_k^{(e)} = 0$$

and is regular at infinity, i.e., of the order $O(1/r)$.

(c) At the surface S it is

$$(3.1.3) \quad U_k^{(i)} = U_k^{(e)}$$

and, if \mathbf{n} is the unit vector of the normal,

$$(3.1.4) \quad (\text{grad } U_k^{(i)} + h_k \text{ grad } U_k^{(e)}) \cdot \mathbf{n} = \frac{\partial U_k^{(i)}}{\partial n} + h_k \frac{\partial U_k^{(e)}}{\partial n} = 0$$

where h_k , $k = 1, 2, 3, \dots$, is a set of positive numbers. To determine these numbers we must have certain conditions. We will assume that these conditions are

$$(3.1.5) \quad J^{(i)} = \int_{\tau_i} (\text{grad } U^{(i)})^2 d\tau \text{ is a minimum}$$

$$(3.1.6) \quad J^{(e)} = \int_{\tau_e} (\text{grad } U^{(e)})^2 d\tau = 1$$

Then, by the equations (3.1.3), (3.1.6), and Green's theorem we obtain

$$J^{(i)} = \int_S U^{(i)} \frac{\partial U^{(i)}}{\partial n} dS, \quad J^{(e)} = - \int_S U^{(e)} \frac{\partial U^{(e)}}{\partial n} dS = - \int_S U^{(i)} \frac{\partial U^{(e)}}{\partial n} dS = 1$$

If we put $U_0^{(i)} = \text{const.}$, it is by (3.1.5) $J_0^{(i)} = 0$. This is a minimum since $J^{(i)} \geq 0$. We assume now that $h_0 = 0$. It is, in general, by (3.1.4)

$$\int_S U_k^{(i)} \left(\frac{\partial U_k^{(i)}}{\partial n} + h_k \frac{\partial U_k^{(e)}}{\partial n} \right) dS = J_k^{(i)} + h_k (-J_k^{(e)}) = 0$$

hence

$$(3.1.7) \quad J_k^{(i)} = h_k$$

The functions U_k which satisfy all the conditions just given are called fundamental functions. The conditions (3.1.7) and (3.1.6) can be also written in the form

$$(3.1.8) \quad \int_S U_k^{(i)} \frac{\partial U_k^{(i)}}{\partial n} dS = h_k, \quad \int_S U_k^{(i)} \frac{\partial U_k^{(e)}}{\partial n} dS = -1.$$

Similar conditions represent connections between functions having different subscripts, namely,

$$(3.1.9) \quad \int_S U_k^{(i)} \frac{\partial U_m^{(i)}}{\partial n} dS = 0, \quad \int_S U_k^{(i)} \frac{\partial U_m^{(e)}}{\partial n} dS = 0 \quad k \neq m$$

The spherical harmonics yield such functions in case of a sphere and the functions of Lamé in case of an ellipsoid.

It has been proved by Poincaré (see also the investigations of Stekloff) that in general if a function Φ is determined at a surface S it can be represented under certain conditions by a series of fundamental functions

$$(3.1.10) \quad \Phi = \sum_{k=1}^{\infty} A_k U_k^{(i)}$$

To find a coefficient A_k we multiply this equation by

$$\frac{\partial U_k^{(e)}}{\partial n} dS$$

and integrate over the surface S . Then, we obtain

$$(3.1.11) \quad A_k = - \int_S \Phi \frac{\partial U_k^{(e)}}{\partial n} dS$$

from (3.1.8) and (3.1.9). This general result has been applied in the theory of figures of equilibrium which differ but little from ellipsoids, and, therefore, we shall now take into account some properties of Lamé functions. Let ϱ , μ , and ν be elliptical coordinates and $R(\varrho^2)$, $S(\varrho^2)$, $M(\mu^2)$, and $N(\nu^2)$ the functions of Lamé. As is well known, the elliptical coordinates correspond to three confocal surfaces passing through a given point $P(x, y, z)$.

The equation of a surface of the second order ($c < b < a$)

$$(3.1.12) \quad \frac{x^2}{\lambda^2 - a^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = 0$$

represents ellipsoids, if $\lambda^2 = \varrho^2$ and $\lambda^2 - a^2 > 0$. It will determine confocal hyperboloids with one nappe if $\lambda^2 = \mu^2$ and $\lambda^2 - a^2 < 0$, $\lambda^2 - b^2 > 0$ and confocal hyperboloids with two nappes, if $b^2 > \lambda^2 = \nu^2 > c^2$.

There are eight points in space corresponding to three given values of ϱ, μ, ν . Let $f(\varrho^2)$ be a polynomial and $n = 2k$ its degree. There are four classes of Lamé functions:

$$(3.1.13) \quad R = f(\varrho^2),$$

$$(3.1.14) \quad R = \sqrt{\varrho^2 - a^2} f(\varrho^2), \sqrt{\varrho^2 - b^2} f(\varrho^2), \sqrt{\varrho^2 - c^2} f(\varrho^2)$$

$$(3.1.15) \quad R = \sqrt{(\varrho^2 - a^2)(\varrho^2 - b^2)} f(\varrho^2), \\ \sqrt{(\varrho^2 - b^2)(\varrho^2 - c^2)} f(\varrho^2), \sqrt{(\varrho^2 - c^2)(\varrho^2 - a^2)} f(\varrho^2)$$

$$(3.1.16) \quad R = \sqrt{(\varrho^2 - a^2)(\varrho^2 - b^2)(\varrho^2 - c^2)} f(\varrho^2)$$

The functions M and N are represented by similar expressions where μ^2 or ν^2 is substituted for ϱ^2 , for example, by (3.1.13) it is $M = f(\mu^2)$, $N = f(\nu^2)$.

Expressions such as MN or RMN are called Lamé products. They are solutions of the Laplace equation. A Lamé function of the second kind S is defined as the second solution of the differential equation

$$(3.1.17) \quad \frac{d^2 R}{du^2} = [n(n+1)\wp u + h]R$$

where $\wp u$ is the Weierstrass function. The constant h in (3.1.17) is determined by the condition that solutions are of the type considered above. The function S must also satisfy the condition

$$(3.1.18) \quad R \frac{dS}{du} - S \frac{dR}{du} = 2n + 1, \quad S = R \int_0^u \frac{2n+1}{R^2} du$$

For more details about the Lamé functions we refer to

Vol. IV, 1, Chapter VI of Appell [1]. We mention here only few particular solutions and values of Lamé products. These particular solutions are as follows:

$$(3.1.13') \quad n = 0, \quad R_0 = 1$$

$$(3.1.14') \quad n = 1, \quad R_1 = \sqrt{\varrho^2 - a^2}, \quad R_2 = \sqrt{\varrho^2 - b^2}, \quad R_3 = \sqrt{\varrho^2 - c^2}$$

$$(3.1.15') \quad n = 2, \quad R_4 = R_2 R_3, \quad R_5 = R_3 R_1, \quad R_6 = R_1 R_2,$$

$$R_7 = \varrho^2 - [g + \frac{1}{6}\sqrt{3g_2}], \quad R_8 = \varrho^2 - [g - \frac{1}{6}\sqrt{3g_2}]$$

if we write the Weierstrass function in the form

$$\wp u = s, \quad \int_{\infty}^s \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} = u$$

and put

$$3g_2 = h^2, \quad Kg + H = h, \quad K = n(n+1),$$

$$\varrho^2 = \wp u + g$$

where H is a constant. It is obtained when the equation (3.1.17) is transformed. Using the variable ϱ this equation takes the form

$$(3.1.17') \quad \frac{A}{R} \frac{d}{d\varrho} \left(A \frac{dR}{d\varrho} \right) = K\varrho^2 + H$$

where

$$A^2 = \frac{(\varrho^2 - a^2)(\varrho^2 - b^2)(\varrho^2 - c^2)}{\varrho^2}$$

For $n = 1$, there are three functions of Lamé and three products which are, denoting by C an arbitrary constant,

$$(3.1.19) \quad RMN = Cx, \quad Cy, \quad Cz$$

For $n = 2$, there are five R functions and, therefore, five Lamé products. Three of them are

$$(3.1.20) \quad RMN = Cxy, \quad Cyz, \quad Czx$$

For $n = 3$, we have seven Lamé products. By (3.1.16) one of these expressions is

$$(3.1.21) \quad RMN = Cxyz$$

A new variable l will be used later. It is defined as a derivative of the variable u introduced in (3.1.17) along the outer normal

$$(3.1.22) \quad \frac{du}{dn_e} = -l, \quad l^2 = \frac{1}{(\mu^2 - \varrho^2)(\nu^2 - \varrho^2)}$$

Now, assume that a certain ellipsoid E_0 is determined by putting $\varrho = \varrho_0$ and that U_0 is a harmonic function on this ellipsoid. We can put according to Poincaré

$$(3.1.23) \quad U_0 = \sum_{k=0}^{\infty} A_k M_k N_k$$

Denoting by the superscript 0 the value of a function for $\varrho = \varrho_0$, we take

$$(3.1.24) \quad A_k = \alpha_k R_k^{(0)} S_k^{(0)}$$

Then, the function

$$(3.1.25) \quad U_i = \sum_{k=0}^{\infty} \alpha_k R_k S_k^{(0)} M_k N_k$$

will be the harmonic function in the interior of E_0 having on this ellipsoid the value U_0 . On the other hand

$$(3.1.26) \quad U_e = \sum_{k=0}^{\infty} \alpha_k R_k^{(0)} S_k M_k N_k$$

is a harmonic function in the space exterior to E_0 and $U_e = U_0$ on E_0 .

Let us assume now that

$$(3.1.27) \quad U_k^{(i)} = \alpha_k S_k^{(0)} R_k M_k N_k, \quad U_k^{(e)} = \alpha_k R_k^{(0)} S_k M_k N_k$$

It is obvious that these functions satisfy the condition $U_k^{(i)} = U_k^{(e)}$ on E_0 . It will also be

$$(3.1.28) \quad \frac{\partial U_k^{(i)}}{\partial n} + h_k \frac{\partial U_k^{(e)}}{\partial n} = 0$$

if we put

$$(3.1.29) \quad h_k = -S_k^{(0)} \left(\frac{\partial R_k}{\partial \varrho} \right)_0 \bigg/ R_k^{(0)} \left(\frac{\partial S_k}{\partial \varrho} \right)_0$$

Thus, U_k are fundamental functions for E_0 because of the properties of Lamé functions.

3.2. Figures of Equilibrium That Differ But Little From Ellipsoids

We shall now find the potential of a homogeneous ellipsoidal layer in terms of Lamé functions. Assume that the varying thickness of it is ζ and its constant density is $\kappa = 1$. The mass element $\zeta d\sigma$ can be interpreted as a product of $d\sigma$ by the surface density ζ . Then, we have the well-known condition for the discontinuity of the force at the simple layer

$$(3.2.1) \quad \frac{\partial U_i}{\partial n_i} + \frac{\partial U_e}{\partial n_e} = -4\pi\zeta$$

On taking into account the expressions (3.1.25), (3.1.26) for the functions U_i and U_e we can write their normal derivatives in the form

$$\begin{aligned} \frac{\partial U_i}{\partial n_i} &= \frac{\partial U_i}{\partial u} \frac{\partial u}{\partial n_i} = \sum_0^\infty \alpha_k S_k^{(0)} \frac{dR_k}{du} M_k N_k \frac{du}{dn_i} = \sum_0^\infty \alpha_0 l_0 S_k^{(0)} \frac{dR_k}{du} M_k N_k \\ \frac{\partial U_e}{\partial n_e} &= \frac{\partial U_e}{\partial u} \frac{\partial u}{\partial n_e} = \sum_0^\infty \alpha_k R_k^{(0)} \frac{dS_k}{du} M_k N_k \frac{du}{dn_e} = - \sum_0^\infty \alpha_k l_0 R_k^{(0)} \frac{dS_k}{du} M_k N_k \end{aligned}$$

where l_0 is the value of l at $\varrho = \varrho_0$ and u the parameter which is introduced in the Lamé differential equation (3.1.17).

Thus, by (3.2.1) we obtain from the expressions for the normal derivatives of U_i and U_e that

$$(3.2.2) \quad -4\pi\zeta = \sum_0^\infty \alpha_k l_0 M_k N_k \left[S_k \frac{dR_k}{du} - R_k \frac{dS_k}{du} \right]_{\rho=\rho_0}$$

By (3.1.18) it is

$$(3.2.3) \quad \zeta = l_0 \sum_0^\infty \frac{2n+1}{4\pi} \alpha_k M_k N_k = l_0 \sum_0^\infty \beta_k M_k N_k$$

where

$$(3.2.4) \quad \beta_k = \frac{2n+1}{4\pi} \alpha_k$$

If

$$(3.2.5) \quad \hat{U} = \sum_0^{\infty} \alpha_k R_k^{(0)} S_k^{(0)} M_k N_k$$

is a given function, the coefficients α_k are known. Then, we have also β_k and can compute coefficients of Lamé products in (3.2.3) and *vice versa*.

If the coefficients C in (3.1.19) are determined to give the connection between cartesian and elliptic coordinates, the former are expressed in terms of latter as follows

$$(3.2.6) \quad x = h_1 R_1 M_1 N_1, \quad y = h_2 R_2 M_2 N_2, \quad z = h_3 R_3 M_3 N_3$$

where

$$(3.2.7) \quad h_1^2 = \frac{1}{(a^2 - b^2)(a^2 - c^2)}, \quad h_2^2 = \frac{1}{(b^2 - a^2)(b^2 - c^2)}, \\ h_3^2 = \frac{1}{(c^2 - a^2)(c^2 - b^2)}$$

The volume of an ellipsoid can be also given in terms of functions R_i . It is, namely,

$$(3.2.8) \quad V = \frac{4\pi}{3} R_1 R_2 R_3 = \frac{4\pi}{3} R_1 R_4 = \frac{4\pi}{3} R_2 R_5 = \frac{4\pi}{3} R_3 R_6$$

Now the direction of the normal to an ellipsoid E is determined by the cosines of the angles as follows

$$(3.2.9) \quad \cos(nx) = \frac{dx}{dn} = h_1 l M_1 N_1 R_4, \quad \cos(ny) = h_2 l M_2 N_2 R_5, \\ \cos(nz) = h_3 l M_3 N_3 R_6$$

If U_E is the potential of a given ellipsoid E and U' that of a layer on this ellipsoid having the thickness ζ , the sum $U_E + U'$ will represent the potential of the deformed ellipsoid. For an incompressible fluid, the thickness of this layer will be subject to the condition

$$(3.2.10) \quad \int_S \zeta d\sigma = 0$$

In order to express the potential of an ellipsoid in terms of Lamé

functions, Poincaré derived first the following formulae for the case where the ellipsoid is displaced in the x -direction (and in a similar manner for other directions). It is by (3.2.9)

$$\zeta = \varepsilon \cos (nx) = \varepsilon h_1 l_0 R_4^0 M_1 N_1$$

If the thickness of the layer is reduced to one term in (3.2.3) its potential is also represented by a single term ($n = 1$)

$$(3.2.11) \quad \frac{4\pi}{3} \varepsilon h_1 R_1^{(0)} S_1^{(0)} R_4^{(0)} M_1 N_1 = U'$$

Now, the change in potential of the ellipsoid due to the displacement ε is obviously

$$(3.2.12) \quad U_{E2} - U_{E1} = U' = -\varepsilon \frac{\partial U_E}{\partial x}$$

since in the second position the same particle has the coordinate $x - \varepsilon$ with respect to point P where the value of the potential is considered. By (3.2.11), (3.2.12), and (3.2.6) the first of the equations (3.2.13) is readily obtained, the other two follow from similar considerations concerning the displacements in the y - or z -direction. Then, each of them can be written in the form:

$$(3.2.13) \quad \frac{\partial U_E}{\partial x} = -\frac{4\pi}{3} R_4^{(0)} S_1^{(0)} x = -V \frac{S_1^{(0)}}{R_1^{(0)}} x,$$

$$\frac{\partial U_E}{\partial y} = -V \frac{S_2^{(0)}}{R_2^{(0)}} y, \quad \frac{\partial U_E}{\partial z} = -V \frac{S_3^{(0)}}{R_3^{(0)}} z$$

It is obvious now that at a point in the interior of the ellipsoid the potential is

$$(3.2.14) \quad U_E = -\frac{V}{2} \left(\frac{S_1^{(0)}}{R_1^{(0)}} x^2 + \frac{S_2^{(0)}}{R_2^{(0)}} y^2 + \frac{S_3^{(0)}}{R_3^{(0)}} z^2 \right)$$

For the external points it may be shown that

$$(3.2.15) \quad \frac{\partial U_E}{\partial x} = -V \frac{S_1}{R_1} x, \quad \frac{\partial U_E}{\partial y} = -V \frac{S_2}{R_2} y, \quad \frac{\partial U_E}{\partial z} = -V \frac{S_3}{R_3} z$$

If we now take into account that, because of notations used in this section, the x -axis corresponds to the small axis of the

ellipsoid and that, therefore, in the equation (1.1.8) x - and z -axis must be interchanged, we shall write $s^2 = y^2 + z^2$ and insert in this equation the expression (3.2.14) for the potential. Then, we obtain, by putting $f = 1$,

$$(3.2.16) \quad -V \frac{S_1^{(0)}}{R_1^{(0)}} x^2 + \left(\omega^2 - V \frac{S_2^{(0)}}{R_2^{(0)}} \right) y^2 + \left(\omega^2 - V \frac{S_3^{(0)}}{R_3^{(0)}} \right) z^2 - \text{const.} = 0.$$

The equation of the ellipsoid $E_0(\varrho = \varrho_0)$ can be also written in the form

$$(3.2.17) \quad f(x, y, z) = \frac{x^2}{R_1^{(0)2}} + \frac{y^2}{R_2^{(0)2}} + \frac{z^2}{R_3^{(0)2}} - 1 = 0$$

by (3.1.12) and by the definition of the functions R_i (3.1.13)–(3.1.15') given earlier. If this ellipsoid is a figure of equilibrium, the equations (3.2.16) and (3.2.17) represent the same surface and we obtain the conditions (on omitting the subscript 0)

$$(3.2.18) \quad \frac{\omega^2}{V} = \frac{S_2 R_2 - S_1 R_1}{R_2^2}$$

and

$$(3.2.19) \quad \frac{S_2 R_2 - S_1 R_1}{R_2^2} = \frac{S_3 R_3 - S_1 R_1}{R_3^2}$$

If the volume of the liquid V and the angular velocity ω are given, these two conditions joined to (3.2.8) will determine the length of axes of the ellipsoid since R and S are functions of a, b, c . In case of the ellipsoids of Maclaurin it is $R_2 = R_3$ and we have only two equations, namely (3.2.18) and (3.2.8). The equation (3.2.19) which is to be used for the ellipsoids of Jacobi, can also be written in the form

$$(S_2 R_3 - R_2 S_3) R_2 R_3 = S_1 R_1 (R_3^2 - R_2^2)$$

and subject to an important transformation. By (3.1.18) we have for $n = 1$

$$3(R_2 R_3)^2 \int_0^u \frac{R_3^2 - R_2^2}{R_2^2 R_3^2} du = S_1 R_1 (R_3^2 - R_2^2)$$

Since it is by (3.1.13)-(3.1.15') $R_3^2 - R_2^2 = b^2 - c^2$ and $R_2 R_3 = R_4$ and by (3.1.18)

$$S_4 = R_4 \int_0^u \frac{5}{R_4^2} d\mathcal{U}$$

we obtain the condition

$$(3.2.20) \quad \frac{R_1 S_1}{3} = \frac{R_4 S_4}{5}$$

It may be shown that this is equivalent to (3.1.19).

Now we shall discuss the existence of figures of equilibrium other than these ellipsoids. On assuming that the new figures differ but little from an ellipsoid of Maclaurin or Jacobi we can represent such a figure (E') as mentioned above as an ellipsoid E covered by a layer having a varying thickness ζ . This may obviously have positive as well as negative values. The condition

$$U_0 + \frac{1}{2}\omega^2(y_0^2 + z_0^2) = \text{const.} = \Pi_0$$

holds for each point P of E , if this ellipsoid is a figure of equilibrium. Therefore, at a point P' in the neighbourhood of P we can write

$$(3.2.21) \quad \Pi = \Pi_0 + \frac{\partial \Pi}{\partial n} \zeta$$

and, if we put

$$(3.2.22) \quad \frac{\partial \Pi}{\partial n} = -g$$

calling g the gravity acceleration, we have

$$(3.2.21') \quad \Pi = \Pi_0 - g\zeta$$

Now, U' is the potential of the ellipsoidal layer and if E' is a figure of equilibrium we have the condition

$$\Pi + U' = \Pi_0 - g\zeta + U' = \text{const.}$$

on this surface, or

$$(3.2.23) \quad U' - g\zeta = \text{const.}$$

The thickness ζ which will determine the deformation or the

departure of the new figure from the ellipsoid can be represented by the series

$$(3.2.3) \quad \zeta = \sum_0^{\infty} l_0 \beta_k M_k N_k$$

If we insert this series in the equation (3.2.12) we obtain

$$\sum_0^{\infty} \beta_k \int_S l_0 M_k N_k d\sigma = 0$$

Because of the properties of Lamé functions (see, for example, Appell [1], IV, p. 159) all these integrals are equal to zero for $k > 0$. Thus, the coefficient β_0 must also vanish and we have

$$(3.2.24) \quad \zeta = \sum_1^{\infty} l_0 \beta_k M_k N_k$$

By (3.2.3)–(3.2.5) we have, then,

$$(3.2.25) \quad U^{(0)} = \sum_1^{\infty} \alpha_k S_k^{(0)} R_k^{(0)} M_k N_k, \quad \alpha_k = \frac{4\pi}{2n+1} \beta_k$$

The value of the gravity acceleration at the pole ($y = 0, z = 0$) is by (3.2.22), (3.2.14), and (3.2.17)

$$(3.2.26) \quad g = V \frac{S_1^{(0)}}{R_1^{(0)}} x = V S_1^{(0)}$$

and

$$(3.2.26') \quad l_0 g = \frac{V S_1^{(0)}}{R_2^{(0)} R_3^{(0)}} = \frac{4\pi}{3} R_1^{(0)} S_1^{(0)}$$

By substituting (3.2.24), (3.2.25), and (3.2.26') in (3.2.23) this condition takes the form

$$(3.2.27) \quad \sum_1^{\infty} 4\pi \beta_k \left(\frac{S_k R_k}{2n+1} - \frac{S_1 R_1}{3} \right) M_k N_k = \text{const.}$$

In this approximation g is considered as a constant. In the equation (3.2.27) the products $M_k N_k$ are variable, but the equation must be satisfied for every point, i.e., for each pair of coordinates μ and ν . Therefore, it must be

$$(3.2.28) \quad \beta_k \left(\frac{S_k R_k}{2n+1} - \frac{S_1 R_1}{3} \right) = 0 \quad k = 1, 2, 3, \dots, \text{const.} = 0$$

These conditions are basic in the theory of Poincaré. They were also given without the factors β_k in the first paper of Liapounov. The factors β_k are coefficients in the series representing the function ζ (3.2.24). They can differ from zero, because of (3.2.28), only if the corresponding second factor vanishes. Taking this fact into account, we have: (a) if $n = 1$ and $k = 1$ the coefficient β_1 can differ from zero, since the second factor in (3.2.28) vanishes identically. This case does not present any interest, the value of β_1 giving just the displacement of the whole ellipsoid in the x -direction. (b) In two cases $n = 1$ and $k = 2$ or $k = 3$, the coefficients β_k must vanish because of (3.2.18) and (3.2.19). (c) If $n = 2$ and $k = 4$, β_4 can differ from zero, since the condition (3.2.20) holds for ellipsoids of Jacobi, but this case is not important either, the value of β_4 just giving a pure rotation of the ellipsoid. Thus, new figures can be expected for $k = 5$ or for larger values of it. In order to answer the question whether there are such new figures of equilibrium Poincaré investigated the more general expression

$$(3.2.29) \quad F = \frac{R_k S_k}{2n+1} - \frac{R_i S_i}{2p+1}$$

where $i \neq k$ and p can be equal to n , and the function

$$(3.2.30) \quad F_1 = \frac{F}{R_k^2} = \frac{1}{2n+1} \frac{S_k}{R_k} - \frac{1}{2p+1} \frac{S_i}{R_i} \left(\frac{R_i}{R_k} \right)^2$$

This function depends on the parameter q^2 varying in the interval (∞, a^2) and the problem is reduced to that of the existence of zeros of F_1 . The detailed discussion of the function (3.2.30) shows that for $n = 2$ and $k = 5, 6$, or 7 , $i = p = 1$ there are no zeros and, therefore, no new figures of equilibrium. In case $k = 8$ we have the branch point for Maclaurin's and Jacobi's ellipsoids.

The first new figures of equilibrium are obtained for $n = 3$ and $n = 4$. In general, in order to find such new figures, i.e.,

the nonvanishing coefficients β_k and, therefore $\zeta \neq 0$, use must be made of Lamé functions of a new kind. These functions can be written in the form

$$(3.2.31) \quad f(\varrho^2), \sqrt{\varrho^2 - b^2} f(\varrho^2), \sqrt{\varrho^2 - c^2} f(\varrho^2), \sqrt{(\varrho^2 - b^2)(\varrho^2 - c^2)} f(\varrho^2)$$

where $f(\varrho^2)$ is an entire rational function of degree k . Its zeros (α, α_i) are between c^2 and a^2 . Let α be the largest root. Then, we can put

$$(3.2.32) \quad f(\varrho^2) = (\varrho^2 - \alpha)(\varrho^2 - \alpha_1) \dots (\varrho^2 - \alpha_{k-1})$$

The number n can be equal to $2k$ or $2k + 1$. We shall write only the equations for two figures of the lowest order.

If $n = 3$, we have the function $R_k = \sqrt{\varrho^2 - c^2}(\varrho^2 - \alpha)$, where $c^2 < \alpha < b^2$. Thus, we can write the expression

$$\zeta = \beta_k l_0 M_k N_k = l_0 \varepsilon R_k M_k N_k$$

The factor ε is equal to $\beta_k / R_k^{(0)}$ and $R_k^{(0)}$ has a constant value on the ellipsoid $\varrho = \varrho_0$. The expressions M_k and N_k are obtained by writing μ and ν instead of ϱ in R_k . Therefore, it is

$$\zeta = l_0 \varepsilon \sqrt{(\varrho^2 - c^2)(\mu^2 - c^2)(\nu^2 - c^2)} (\varrho^2 - \alpha)(\mu^2 - \alpha)(\nu^2 - \alpha)$$

Now, by (3.2.6) this square root is proportional to z and

$$(\varrho^2 - \alpha)(\mu^2 - \alpha)(\nu^2 - \alpha) = C \left(\frac{x^2}{\alpha - a^2} + \frac{y^2}{\alpha - b^2} + \frac{z^2}{\alpha - c^2} - 1 \right)$$

both expressions having the same roots because of (3.1.12). Therefore, the equation of the new figure of equilibrium takes the form

$$(3.2.33) \quad \frac{\zeta}{l_0} = \varepsilon' z \left(\frac{x^2}{\alpha - a^2} + \frac{y^2}{\alpha - b^2} + \frac{z^2}{\alpha - c^2} - 1 \right)$$

where $\varepsilon' = \text{const.}$ This is the so-called ovoidal or pear-shaped figure.

In the case $n = 4$, we put $R_k = f(\varrho^2) = (\varrho^2 - \alpha_1)(\varrho^2 - \alpha_2)$ where $c^2 < \alpha_2, \alpha_1 < b^2$. It is now

$$\zeta = l_0 \varepsilon (\varrho^2 - \alpha_1)(\mu^2 - \alpha_1)(\nu^2 - \alpha_1)(\varrho^2 - \alpha_2)(\mu^2 - \alpha_2)(\nu^2 - \alpha_2)$$

and this equation can be written in the form

$$(3.2.34) \quad \frac{\zeta}{l_0} = \varepsilon' \left(\frac{x^2}{\alpha_1 - a^2} + \frac{y^2}{\alpha_1 - b^2} + \frac{z^2}{\alpha_1 - c^2} - 1 \right) \\ \left(\frac{x^2}{\alpha_2 - a^2} + \frac{y^2}{\alpha_2 - b^2} + \frac{z^2}{\alpha_2 - c^2} - 1 \right)$$

The constant ε' can be either positive or negative, and, therefore, two new figures are determined by this equation.

These examples and the general analysis show that there are in general in the expression of ζ as many factors of the form

$$E_i \equiv \frac{x^2}{\alpha_i - a^2} + \frac{y^2}{\alpha_i - b^2} + \frac{z^2}{\alpha_i - c^2} - 1$$

as there are zeros of the function R_k . Thus, it is, in general,

$$(3.2.35) \quad \begin{aligned} \zeta &= l_0 \varepsilon' E_1 E_2 \dots E_k \quad \text{if } n = 2k \\ \zeta &= l_0 \varepsilon' z E_1 E_2 \dots E_k \quad n = 2k + 1 \end{aligned}$$

The conditions here discussed hold for the ellipsoids of Jacobi. The conditions of the existence of new figures of equilibrium which

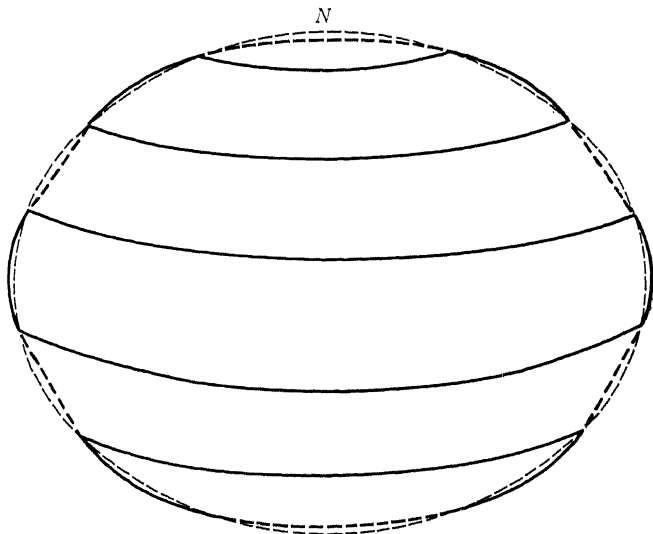


Figure 2

differ but little from the ellipsoids of Maclaurin take a different form. We must have first $n + k \equiv 0 \pmod{2}$, i.e., n and k must be both either even or odd numbers. Putting

$$(3.2.36) \quad \frac{z}{y} = \tan \psi, \sqrt{\frac{a^2 - \mu^2}{a^2 - c^2}} = \cos \vartheta = \bar{\mu}$$

we can make use of the fact that the Lamé products $M_k N_k$, in case of ellipsoids of revolution, determine the spherical harmonics. Taking these functions in the form (1.6.9) we can write

$$(3.2.37) \quad \zeta = l_0 (\beta_k P_{n,k}(\bar{\mu}) \cos k\psi + \beta_{k'} P_{n,k}(\bar{\mu}) \sin k\psi)$$

If we put $\beta_{k'} = \beta_k \tan \bar{\omega}$ this equation takes the form

$$(3.2.38) \quad \frac{\zeta}{l_0} = \frac{\beta_k}{\cos \bar{\omega}} P_{n,k}(\bar{\mu}) \cos(k\psi - \bar{\omega})$$

It may be easily seen that the axes of coordinates can be rotated to make the angle $\bar{\omega} = 0$. Then, (3.2.38) takes the form

$$(3.2.39) \quad \zeta = l_0 \beta_k P_{n,k}(\bar{\mu}) \cos k\psi$$

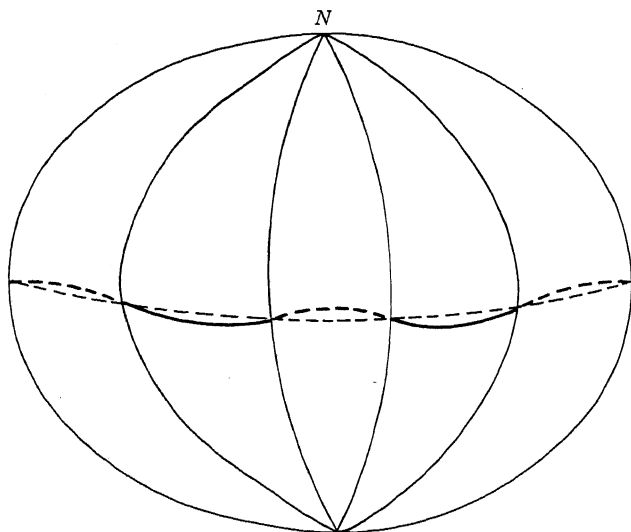


Figure 3

and for $0 < k < n$ it will determine a set of new figures of equilibrium differing but little from the ellipsoids of Maclaurin.

In two extreme cases $k = 0$ and $k = n$ we have

$$(3.2.40) \quad \zeta = l_0 \beta_0 P_n(\bar{\mu}) \quad P_n = \frac{1}{2 \cdot 4 \cdot \dots \cdot 2n} \frac{d^n(\bar{\mu}^2 - 1)^n}{d\bar{\mu}^n}$$

and

$$(3.2.41) \quad \zeta = l_0 \beta_n P_{n,n}(\bar{\mu}) \cos n\varphi, \quad P_{n,n} = (1 - \bar{\mu}^2)^{n/2} \frac{d^n P_n}{d\bar{\mu}^n}$$

Equation (3.2.40) represents a "zonal" figure of equilibrium. The function ζ does not vanish identically at the equator or at the poles. A typical figure of this kind is shown in Figure 2. Equation (3.2.41) gives the so-called sectorial figures: ζ vanishing at poles and along a set of meridians (Figure 3). If $0 < k < n$ we obtain a network of meridians and parallels dividing the surface of the ellipsoid into cells where ζ is alternating from positive to negative values.

3.3. Stability of Figures of Equilibrium

Since the theory of figures of equilibrium also had the purpose of explaining the evolution of celestial bodies, the stability of different figures had to be determined. This question, however, is one of the most difficult in the whole theory. It requires sometimes, as it was the case of the pear-shaped figure, a very intricate analysis and the highest precision is necessary in the notions and criteria used.

Forces of three types can be considered according to the general discussion of the equilibrium given by Thomson and Tait [1]. First, there are forces usually taken into account like universal attraction which depend only on the coordinates of particles. They represent a system of conservative forces. Now, if a liquid mass is in equilibrium, the corresponding equations are the same for an ideal or a viscous liquid. However, if it is assumed that a viscous mass is deformed, the new state differing (maybe even little) from an equilibrium will in general give rise to the second type of forces which depend on velocities such as friction. In

this case, the system is a dissipative one. For most of the figures that are considered, the equilibrium is defined with respect to a set of moving axes, and the third type of forces is represented by the centrifugal and Coriolis force. For these forces Thomson and Tait used the name gyroscopical forces. The basic statements concerning the influence of introduction into conditions of equilibrium of a new type of forces are as follows.

An unstable equilibrium produced by conservative forces alone can be transformed under certain conditions into a stable one if the gyroscopical forces will be also introduced.

The kind of equilibrium which exists under the action of conservative forces alone will not be changed if the forces of both other types will be introduced.

If the equilibrium is stable when forces of all three types are acting, the removal of dissipative forces as well as of the gyroscopical ones does not affect the type of equilibrium; similarly the instability will also persist.

These conclusions follow from equations of motion of a system of particles, written, for example, in the Lagrange form and from the fundamental condition

$$(3.3.1) \quad \frac{d}{dt}(T - U) = -f < 0$$

In this condition $2T$ represents, as usual, the kinetic energy of the system, U is the potential of conservative forces, i.e., $F = \text{grad } U$ and $2f = b_{11}q_1^2 + \dots + b_{nn}q_n^2$ where the coefficients b_{11} are not negative is the dissipation function. The condition (3.3.1) shows that the total energy of the material system is dissipated if the function f differs from zero.

If in the expression for f , all coordinates are involved the dissipative forces are *complete*. According to Lord Kelvin the stability is a *secular* one if it takes place under the action of complete dissipative forces. It is the *temporary* stability if there are incomplete dissipative forces acting. For further development of the theory, certain criteria of stability were introduced in mathematical form.

It is well known that if the equations of motion of a mass system are written in the Lagrange form

$$(3.3.2) \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i}$$

the positions of equilibrium are obtained from the equations

$$(3.3.3) \quad \frac{\partial U}{\partial q_i} = 0 \quad i = 1, 2, \dots, n$$

The classical investigations have proved that U reaching its maximum is the necessary and sufficient condition of a stable equilibrium. It is, of course, a new assumption that the results proved for a discrete mass system hold for a continuous body as well. Since it is made in the theory of figures of equilibrium of fluid masses, this assumption enables us to make an important conclusion from the equation (1.1.8).

$$(3.3.4) \quad \int_V \frac{\kappa' dV'}{D} + \frac{\omega^2 s^2}{2f} = \text{const.}$$

If we multiply the expression representing the left-hand member by $f \kappa dV$ and integrate over the whole volume of the fluid we obtain

$$(3.3.5) \quad W + \frac{1}{2} I \omega^2 = f \int \kappa dV \int \frac{\kappa' dV'}{D} + \frac{\omega^2}{2} \int s^2 \kappa dV$$

where W is the energy of the system due to attraction and I the moment of inertia with respect to the axis of rotation. It is easy to see that in the case of equilibrium this expression is constant, and, therefore, its derivatives with respect to the coordinates are equal to zero. In general, in small movements which will follow, the distortion of a figure of equilibrium forces of all three types will be involved. Lord Kelvin's criterion of stability is, then, that

$$(3.3.6) \quad W + \frac{1}{2} I \omega^2$$

is a maximum.

In this expression the angular velocity is assumed to have a given value. There are, however, cases in which a deformation of a figure of equilibrium will not be performed under the action of

dissipative forces. Poincaré indicated a uniform contraction of a figure of equilibrium as an example of such a phenomenon. In his opinion, to assume that the total momentum is constant in such a case is more natural than to put $\omega = \text{const}$. This momentum being $\tilde{M} = I\omega$ Poincaré proved the validity of another criterion of stability. This new necessary and sufficient condition of stability of equilibrium is that

$$(3.3.7) \quad W - \frac{1}{2} \frac{\tilde{M}}{I}$$

is a maximum. It does not follow from (3.3.6).

Now, either ω or M or some other quantity can be considered as a varying parameter. Then, we obtain a continuous set of figures of equilibrium depending on this parameter. This representation is

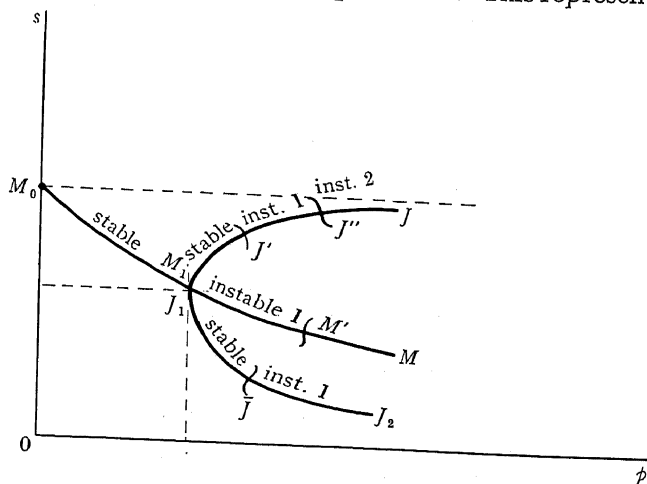


Figure 4

discussed in all details in Appell [1]. We shall mention here only the diagram corresponding to the distribution of stable and unstable figures of equilibrium. In this diagram (Figure 4) a parameter p is used which is proportional to the angular momentum of the liquid mass and $s = c^2/a^2$, where c and a are semi-axes. Then, the set of Maclaurin's ellipsoids is represented by the line $M_0 M_1 M$.

It starts at a point M_0 which corresponds to a sphere and on the branch $M_0 M_1$ we have stable Maclaurin's ellipsoids. These become unstable for values of parameters giving the part $M_1 M$. At the branch point M_1 the stability is transferred to Jacobi's ellipsoids the set of which is represented by the line $J J_1 J_2$. As mentioned in Section (2.3), the new figures of equilibrium differing but little from ellipsoids are determined by the zeros of expressions (2.3.29) or (2.3.30). Therefore, the next branch points on each of the lines $M_0 M_1 M$ or $J J_1 J_2$ will be determined by the roots of equations

$$(3.3.8) \quad \frac{R_k S_k}{2n+1} - \frac{R_1 S_1}{3} = 0$$

and

$$(3.3.9) \quad R_2 = R_3 \quad \text{or} \quad \frac{R_4 S_4}{5} - \frac{R_1 S_1}{3} = 0$$

the former holding for $M_1 M$, the latter for $J J_1 J_2$.

At each branch point, when moving from the first one (M_1) the degree of instability of the ellipsoids belonging to the same set is increasing. It is obvious that all these considerations present a very high interest, especially from the viewpoint of Rational Mechanics, but cannot be applied without restrictions to the evolution of figures of planets or stars, since the assumption that celestial bodies are homogeneous masses is an oversimplification.

3.4. The Ovoidal Figure of Equilibrium

Among the figures of equilibrium of a homogeneous liquid mass differing but little from an ellipsoid there is one which attracted a special attention of investigators. It is given by the equation (3.2.33)

$$(3.4.1) \quad \frac{\xi}{l_0} = \varepsilon' z \left(\frac{x^2}{\alpha - a^2} + \frac{y^2}{\alpha - b^2} + \frac{z^2}{\alpha - c^2} - 1 \right)$$

and called a pear-shaped or ovoidal figure. If

$$(3.4.2) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is the critical ellipsoid of Jacobi, by deformation of which the ovoidal figure can be obtained and $\tilde{x}, \tilde{y}, \tilde{z}$ are coordinates of a point at this surface, we can write

$$(3.4.3) \quad \zeta^2 = (\tilde{x}-x)^2 + (\tilde{y}-y)^2 + (\tilde{z}-z)^2$$

If $(x/\tilde{a}^2)/l_0, \dots$ are the cosines of angles which the normal to (3.4.2) makes with the axes of reference, we have

$$(3.4.4) \quad \tilde{x} = x + \frac{x}{\tilde{a}^2} l_0 \zeta, \quad \tilde{y} = y + \frac{y}{\tilde{b}^2} l_0 \zeta, \quad \tilde{z} = z + \frac{z}{\tilde{c}^2} l_0 \zeta$$

The intersection of the surface (3.4.1) with the Jacobi ellipsoid is indicated in Figure 5. It was pointed out by P. Humbert [1] that the meridians of the ovoidal figure do not have points of inflexion.

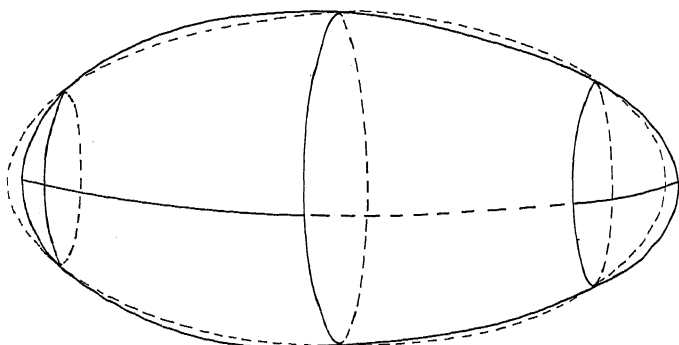


Figure 5

Therefore, this figure is actually more like an egg than a pear. The representation given by Poincaré [2] has been based on the first approximation, and this suggested the name pear-shaped. Certain speculations concerning the fission of a celestial mass have been made because of this representation. In a theory of formation of the Moon developed by Darwin [3], [4], p. 317, the possibility of such a process was connected with the question of stability of the pear-shaped figure. If a figure having such shape were stable, an evolution of it could be expected such that the narrow belt would contract, and the figure would be divided into two separate

masses. The first investigations of Liapounov have shown, however, that the ovoidal figure is unstable and, therefore, the separation of the Moon from the Earth should not be explained by the fact of the existence of such a figure. The question of stability of the ovoidal figure has been considered as an open one for a longer time (see, Appell [1]). Upon developing a new method of approach, Jeans [1], [3] confirmed Liapounov's conclusion.

In a recent discussion of the question of stability Lyttleton [1] corrected several other statements of Jeans concerning the fission of a rotating fluid mass. He reached the conclusion that the fission process as applied by Jeans to the origin of binary stars is also impossible.

3.5. Some Other Figures of Equilibrium

To explain the existence of the ring of Saturn and the shape of some nebulae, the ring-shaped figures of equilibrium were investigated. A critical analysis of the results concerning the figures of this kind was given by Poincaré [2]. Although the study of a ring-shaped figure of equilibrium of a liquid mass was initiated by Laplace, an exact proof of the existence of such a figure has been given much later by Sophie Kovalevsky [1]. However, neither the hypothesis of a liquid ring, nor that of a solid can explain the ring of Saturn as it was shown in investigations of Laplace, Maxwell, and Poincaré. It was mentioned before that only the hypothesis of Cassini (that the ring of Saturn is composed of small solid particles) can meet all requirements of observations and theory.

Some other figures of equilibrium were obtained in the two-dimensional problem. These figures investigated by Matthiessen in 1859, Jeans [1], and Globa-Mikhailenko [1] are infinite cylinders with an elliptical cross section. It was pointed out by Liapounov that such problems are nothing but mathematical curiosities.

Method of Liapounov

4.1. Potential of a Body in Which the Stratification Differs But Little From an Ellipsoidal One

The solution of a problem concerning the figures of equilibrium of a fluid mass depends on the series expansion of the potential of this mass. As mentioned previously, Clairaut's theory could not be extended over the first approximation. Although the methods of Laplace and Legendre yielded any desired approximation in a formal way, the convergence of the series used remained an open question. Neither Poisson nor Callandreau could make essential progress investigating it. Even the degree of precision which could be obtained by the method of Poincaré (see Chapter III) was not quite satisfactory as pointed out by Liapounov. In a series of papers he developed, therefore, a new method of attacking the problem. After more than 30 years of investigation in this field, the final form of the method was given by Liapounov in his last work [9] published 1925, seven years after the tragic death of its author. This form of the method which can be applied to a heterogeneous body as well as to a homogeneous one will be called here the method of Liapounov.

It must be pointed out that the conclusions are not restricted by the assumption that the departures from the ellipsoidal structure which will be taken into account, are infinitely small. They can also have finite values, and in this respect as well the method of Liapounov is superior to other methods.

Now, in order to write the expression for the potential of a body the stratification in which differs but little from that which is given by a set of surfaces of ellipsoids, we shall mostly use the notations of Liapounov. Let $E(\sqrt{\varrho+1}, \sqrt{\varrho+q}, \sqrt{\varrho})$ be an

ellipsoid that belongs either to the set of Maclaurin's ellipsoids ($q = 1$) or to that of ellipsoids of Jacobi ($q < 1$). Then $E_a(a\sqrt{q+1}, a\sqrt{q+q}, a\sqrt{q})$ represents a set of ellipsoids similar and concentric to E . Suppose that in a given body B the surfaces of equal density differ but little from the ellipsoids E_a and that the density is decreasing from the center of the body towards its free surface. We assume that the variation of the density is small and that the law of densities is given in the form

$$(4.1.1) \quad \kappa = 1 + \delta\varphi(a)$$

where φ is a given function of a parameter a . This parameter will determine the surfaces of equal density ($\kappa = \text{const.}$); δ is a second parameter having small values. The case of a homogeneous body is obtained from the formulae which follow on putting $\delta = 0$, $\kappa = \text{const.}$ Obviously the parameter a can be given different interpretations. We shall assume that a is determined by the condition that the volume of the space bounded by the level surface $a = a_k$ is equal to that of the ellipsoid E_{a_k} . In the center of the body it is $a = 0$, and at the free surface F of B we have $a = 1$.

If $\varphi(0)$ is zero the density at the center is $\kappa_0 = 1$, i.e., the value adopted by Emden [1] for a standard solution. The form (4.1.1) was chosen by Liapounov to make easier the application of the series expansion obtained for a homogeneous ellipsoid in the problem concerning heterogeneous mass as will be seen later (equation (4.3.4)). In general, however, we could assume that

$$(4.1.1') \quad \kappa = \varphi(a, \delta)$$

and, on expanding this function in a power series with respect to the parameter δ obtain

$$(4.1.1'') \quad \kappa = 1 + \sum \delta^k \varphi_k(a)$$

The results of Liapounov's investigation can be easily generalized for this law of densities on adding terms corresponding to the sum

$$\sum_{k=2}^{\infty} \delta^k \varphi_k.$$

Now, according to Liapounov, the level surfaces F_a can be

represented by the equations

$$(4.1.2) \quad \begin{aligned} x &= a(1 + \zeta) \sqrt{\varrho + 1} \sin \vartheta \cos \psi \\ y &= a(1 + \zeta) \sqrt{\varrho + q} \sin \vartheta \sin \psi \\ z &= a(1 + \zeta) \sqrt{\varrho} \cos \vartheta \end{aligned}$$

where ϑ and ψ are spherical coordinates. The function ζ must have small values according to assumptions just made. In general, the formulae (4.1.2) will determine the position of any point in space if $0 \leq a \leq \infty$, $0 \leq \vartheta \leq \pi$, $0 \leq \psi \leq 2\pi$. To each function $\zeta(a, \vartheta, \psi)$ corresponds a set of surfaces F_a . On the free surface F it is

$$(4.1.3) \quad \zeta(1, \vartheta, \psi) = \bar{\zeta}(\vartheta, \psi) = \bar{\zeta}$$

If we assume that to each pair of values of the angles ϑ and ψ corresponds a single value of ζ and that the differential coefficient $\partial\zeta/\partial a$ does exist, the equations (4.1.2) yield the expression for a volume element as follows

$$(4.1.4) \quad dV = a^2 \sqrt{\varrho(\varrho + q)(\varrho + 1)} (1 + \zeta)^2 \left(1 + \frac{\partial a \zeta}{\partial a}\right) \sin \vartheta \, da \, d\vartheta \, d\psi$$

We put now

$$(4.1.5) \quad \Delta = \sqrt{\varrho(\varrho + q)(\varrho + 1)} \quad d\sigma = \sin \vartheta \, d\vartheta \, d\psi$$

Since according to the definition of the parameter a the volumes E_a and F_a must be equal we have

$$\begin{aligned} \Delta \int_0^a a^2 \, da \int (1 + \zeta)^2 \left(1 + \frac{\partial a \zeta}{\partial a}\right) d\sigma \\ = \frac{\Delta}{3} a^3 \int (1 + \zeta)^3 d\sigma = \frac{4}{3} \pi a^3 \sqrt{\varrho(\varrho + q)(\varrho + 1)} \end{aligned}$$

By (4.1.5) it follows from this equation that $\int (1 + \zeta)^3 d\sigma = 4\pi$ and, therefore, we obtain the first condition for the function ζ in the form

$$(4.1.6) \quad \int \zeta \, d\sigma = - \int \zeta^2 \, d\sigma - \frac{1}{3} \int \zeta^3 \, d\sigma$$

Let U be the potential at a point $M(x, y, z)$. At $M'(x', y', z')$ we have

$$U = \int \frac{\kappa' dV'}{D}$$

where $D = MM'$. On substituting (4.1.4) and (4.1.5) we obtain

$$(4.1.7) \quad U = \Delta \int_0^1 \kappa' a'^2 da' \int \frac{(1 + \zeta')^2 \left(1 + \frac{\partial a' \zeta'}{\partial a'}\right) d\sigma'}{D}$$

In order to solve the problem it will be necessary to expand this potential in a series. Then, the function ζ must be subjected to certain conditions. Liapounov assumes first that this function can be expanded in a power series with respect to the parameter δ . We shall see that in some other problems, use can be made of certain other parameters as well. Let $l(\delta)$ and $g(\delta)$ be two variable numbers that do not depend on $a, a', \vartheta, \vartheta', \psi, \psi'$ and vanish for $\delta = 0$. We put

$$(4.1.8) \quad (D)_{\zeta=\zeta'=0} = D_0$$

Then two other conditions for ζ introduced by Liapounov are

$$(4.1.9) \quad |\zeta| < l, \quad \frac{\sqrt{e+1}(a+a')}{2} \frac{|\zeta' - \zeta|}{D_0} < g$$

Since ζ is not a given function, the existence of two such numbers should be proved and this has been done by Liapounov after a formal evaluation of ζ .

For further transformations it is convenient to take a new variable ξ instead of ζ' using the definition

$$(4.1.10) \quad \frac{\zeta' - \zeta}{1 + \zeta} = \xi$$

Hence

$$(4.1.11) \quad \frac{1 + \zeta'}{1 + \zeta} = 1 + \xi \quad (1 + \zeta)^{-1} \left(1 + \frac{\partial a' \zeta'}{\partial a'}\right) = 1 + \frac{\partial a' \xi}{\partial a'}$$

By (4.1.2) we obtain

$$\begin{aligned}
D &= \sqrt{(x-x')^2 + \dots} \\
&= \sqrt{(\varrho+1)[a(1+\zeta) \sin \vartheta \cos \psi - a'(1+\zeta') \sin \vartheta' \cos \psi']^2 + \dots} \\
&= (1+\zeta) \sqrt{(\varrho+1)[a \sin \vartheta \cos \psi - a'(1+\xi) \sin \vartheta' \cos \psi']^2 + (\varrho+q)[\dots]^2 + \varrho[\dots]^2}
\end{aligned}$$

If we put $a'(1+\xi) = b$ and $D(a, b)$ for the second factor in (4.1.12) and use new notations we obtain

$$\begin{aligned}
(4.1.12') \quad D &= (1+\zeta)D(a, b) = D(a + a\zeta, a' + a'\zeta') = D[\zeta, \zeta'] \\
D(a, a') &= D[0, 0] = D_0
\end{aligned}$$

Expressed in terms of the new variable ξ the potential U takes the form

$$\begin{aligned}
(4.1.13) \quad U &= \Delta(1+\zeta)^2 \int_0^1 \kappa' da' \int \frac{a'^2(1+\xi)^2 \left(1 + \frac{\partial a' \xi}{\partial a'}\right) d\sigma'}{D(a, b)} \\
&= \Delta(1+\zeta)^2 \int_0^1 \kappa' da' \int \left(1 + \frac{\partial a' \xi}{\partial a'}\right) \frac{b^2 d\sigma'}{D(a, b)}
\end{aligned}$$

To expand now the surface integral here involved in a power series we will take into account that

$$\int \frac{b^2 d\sigma'}{D(a, b)} = \sum_{n=0}^{\infty} \frac{a'^n}{n!} \left\{ \frac{\partial^n}{\partial b^n} \int \frac{b^2 \xi^n d\sigma'}{D(a, b)} \right\}_{b=a'}$$

since $b = a' + a'\xi$, and that

$$\int \frac{b^2 \frac{\partial a' \xi}{\partial a'}}{D(a, b)} d\sigma' = \sum_{n=0}^{\infty} \frac{a'^2}{n!} \left\{ \frac{\partial^n}{\partial b^n} \int \frac{b^2 \xi^n \frac{\partial a' \xi}{\partial a'}}{D(a, b)} d\sigma' \right\}_{b=a'}$$

It is

$$\frac{a'^n \xi^n}{n!} \frac{\partial a' \xi}{\partial a'} = \frac{1}{(n+1)!} \frac{\partial (a' \xi)^{n+1}}{\partial a'}$$

and, therefore, the surface integral in (4.1.13) is

$$\begin{aligned}
J &= \int \frac{b^2 \left(1 + \frac{\partial a' \xi}{\partial a'}\right)}{D(a, b)} d\sigma' = \int \frac{a'^2 d\sigma'}{D(a, a')} + \sum_{n=1}^{\infty} \left\{ \frac{\partial^{n-1}}{\partial b^{n-1}} \frac{\partial}{\partial b} \int \frac{a'^n \xi^n}{n!} \frac{b^2 d\sigma'}{D(a, b)} \right\}_{b=a'} \\
&+ \sum_{n=1}^{\infty} \left\{ \frac{\partial^{n-1}}{\partial b^{n-1}} \frac{\partial}{\partial a'} \int \frac{a'^n \xi^n}{n!} \frac{b^2 d\sigma'}{D(a, b)} \right\}_{b=a'} \\
&= \int \frac{a'^2 d\sigma'}{D(a, a')} + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial}{\partial a'} \left\{ \frac{\partial^{n-1}}{\partial b^{n-1}} \int \frac{a'^n \xi^n b^2 d\sigma'}{D(a, b)} \right\}_{b=a'}
\end{aligned}$$

On substituting the variable ζ' again this expression takes the form

$$\begin{aligned}
J &= \int \frac{a'^2 d\sigma'}{D(a, a')} + \frac{1}{1 + \zeta} \int \frac{a'^3 (\zeta' - \zeta) d\sigma'}{D(a, a')} \\
&+ \sum_{n=2}^{\infty} \frac{1}{(1 + \zeta)^n} \frac{1}{n!} \frac{\partial}{\partial a'} \left\{ \frac{\partial^{n-1}}{\partial b^{n-1}} \int \frac{a'^n (\zeta' - \zeta)^n b^2 d\sigma'}{D(a, b)} \right\}_{b=a'}
\end{aligned}$$

Thus, (4.1.13) is transformed into equation

$$\begin{aligned}
(4.1.14) \quad U &= (1 + \zeta)^2 \Delta \int_0^1 \kappa' da' \int \frac{a'^2 d\sigma'}{D(a, a')} \\
&+ (1 + \zeta) \Delta \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \int \frac{a'^3 (\zeta' - \zeta) d\sigma'}{D(a, a')} \\
&+ \Delta \sum_{n=2}^{\infty} \frac{(1 + \zeta)^{2-n}}{n!} \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \left\{ \frac{\partial^{n-1}}{\partial b^{n-1}} \int \frac{a'^n (\zeta' - \zeta)^n b^2 d\sigma'}{D(a, b)} \right\}_{b=a'}
\end{aligned}$$

and we can put

$$(4.1.15) \quad U = U_0 + U_1 + U_2 + \dots$$

The terms of this series are now given by the expressions

$$\begin{aligned}
(4.1.16) \quad U_0 &= \Delta \int_0^1 a'^2 \kappa' da' \int \frac{d\sigma'}{D(a, a')} \\
U_1 &= 2\zeta U_0 + \Delta \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \int \frac{a'^3 (\zeta' - \zeta)}{D(a, a')} d\sigma' \\
U_2 &= \zeta^2 U_0 + \zeta \Delta \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \int \frac{a'^3 (\zeta' - \zeta)}{D(a, a')} d\sigma' \\
&+ \frac{\Delta}{2} \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \left\{ \frac{\partial}{\partial b} \int \frac{a'^2 b^2 (\zeta' - \zeta)^2}{D(a, b)} d\sigma' \right\}_{b=a'}
\end{aligned}$$

The values of these terms at the ellipsoid's center ($a = 0$) are

$$(4.1.17) \quad \begin{aligned} U_0(0) &= \Delta \int_0^1 \kappa' a'^2 da' \int \frac{d\sigma'}{D(0, a')} \\ U_1(0) &= \Delta \int_0^1 \kappa' da' \frac{\partial}{\partial a'} \int \frac{a'^3 \xi' d\sigma'}{D(0, a')} \end{aligned}$$

For $n > 2$ it is $U_n(0) = 0$. On making use of the method of Cauchy (majorantes functions), Liapounov was able to prove that (4.1.15) is an absolutely and uniformly convergent series, if the conditions (4.1.9) are satisfied and if

$$(4.1.18) \quad l + g < 1$$

The first term in (4.1.15) U_0 is the potential of the ellipsoid E in which the ellipsoids E_a are surfaces of equal density. This can be easily seen from (4.1.2) and (4.1.7) where one has to put $\xi = \xi' = 0$.

Under conditions (4.1.9) and (4.1.18) the expression

$$\frac{U' - \hat{U}}{D}$$

can be also expanded in an absolutely and uniformly convergent series, as it has been shown by Liapounov, and it is very important to note that in all these expansions ξ as well as δ are not assumed to be infinitely small. The formulae just developed will show how the functional equation (1.1.8) can be reduced to a set of integral and integro-differential equations.

4.2. Integral Equation of Liapounov

The further development of the theory is based on the solution of the integral equation

$$(4.2.1) \quad \xi - \nu \int_x \frac{\xi' d\sigma'}{D} = \mathcal{E}$$

which has been given by Liapounov. In this equation ξ is a function of two spherical coordinates ϑ and ψ and \mathcal{E} is a given continuous function of these variables. We assume that \mathcal{E} depends only

on ϑ and ψ . D is, as before, the distance between the points $M(\vartheta, \psi)$ and $M'(\vartheta', \psi')$ of the ellipsoid E , and the integration is over the unit sphere Σ . Putting in (4.1.12) $\xi = \xi' = 0$, $a = a' = 1$ we obtain

$$(4.2.2) \quad D^2 = (\varrho + 1)[\sin \vartheta \cos \psi - \sin \vartheta' \cos \psi']^2 \\ + (\varrho + q)[\sin \vartheta \sin \psi - \sin \vartheta' \sin \psi']^2 + \varrho[\cos \vartheta - \cos \vartheta']^2$$

Instead of this expression, the distance $D(a, 1)$ can be also used in the equation (4.2.1) which is obtained from (4.1.12) and (4.1.12') if $\xi = \xi' = 0$ and $a' = 1$.

In order to solve the inhomogeneous linear-integral equation (4.2.1) use has to be made of Liouville's homogeneous equation (Section (1.6)).

If the function Ξ is continuous ξ will also be continuous. Now, equation (4.2.1) will have a solution if the function will satisfy certain conditions. To obtain these conditions we multiply (4.2.1) by $Y_{n,s} d\sigma$ and integrate the product over the unit sphere. Then we obtain

$$\int \xi Y_{n,s} d\sigma - \nu \int Y_{n,s} d\sigma \int \frac{\xi' d\sigma'}{D} = \int \Xi Y_{n,s} d\sigma$$

Reversing the integration in the second term it is

$$- \nu \int \xi' d\sigma' \int \frac{Y_{n,s} d\sigma}{D}$$

Since the expression (4.2.2) is symmetrical with respect to the variables, the formulae (1.6.5)–(1.6.6) yield

$$- \nu \int \xi' \frac{Y'_{n,s}}{\nu_{n,s}} d\sigma' = - \frac{\nu}{\nu_{n,s}} \int \xi Y_{n,s} d\sigma$$

Hence

$$\frac{\nu_{n,s} - \nu}{\nu_{n,s}} \int \xi Y_{n,s} d\sigma = \int \Xi Y_{n,s} d\sigma$$

From (1.6.5) and (1.6.10) it follows that the expression

$$(4.2.3) \quad T_{n,s} = \frac{\nu_{n,s} - \nu}{\nu_{n,s}}$$

depends on ϱ . Then, we have

$$(4.2.4) \quad T_{n,s} = 0 \quad \text{if } \nu = \nu_{n,s}$$

i.e., for certain values of ϱ only. Ellipsoids which correspond to these values are the ellipsoids of bifurcation. For the ellipsoids of Maclaurin there are conditions

$$(4.2.5) \quad T_{1,0} = 0 \quad \nu_{1,0} = \nu$$

and

$$(4.2.6) \quad T_{p,2k} = 0 \quad T_{p,2k-1} = 0$$

Another set of conditions is required for the ellipsoids of Jacobi. Humbert [1] and Orlov [2] calculated the elements of ellipsoids of bifurcation or critical ellipsoids up to those of the seventh order.

Now, taking into account (4.2.3) and the equation preceding it we can easily see that the function \mathcal{E} must satisfy the conditions

$$(4.2.7) \quad \int \mathcal{E} Y_{n,s} d\sigma = 0$$

for all values of n and s such that the equations (4.2.4) hold. It is always

$$(4.2.8) \quad \int \mathcal{E} Y_{1,0} d\sigma = 0$$

We assume now that \mathcal{E} can be expanded in a Laplace series

$$(4.2.9) \quad \mathcal{E} = \sum A_{n,s} Y_{n,s}$$

which converges absolutely and uniformly. Then, it is

$$(4.2.10) \quad A_{n,s} = \frac{1}{\gamma_{n,s}} \int \mathcal{E} Y_{n,s} d\sigma$$

where

$$(4.2.11) \quad \gamma_{n,s} = \int Y_{n,s}^2 d\sigma$$

since the functions $Y_{n,s}$ are orthogonal. A particular solution of (4.2.1) can be readily found. It is

$$(4.2.12) \quad \xi = \sum \frac{A_{n,s}}{T_{n,s}} Y_{n,s}$$

where the summation is extended over all values of n and s except those for which it is $T_{n,s} = 0$. Substituting now (4.2.9) and (4.2.12) into equation (4.2.1) we obtain

$$\sum \left(\frac{A_{n,s}}{T_{n,s}} Y_{n,s} - \frac{\nu A_{n,s}}{T_{n,s}} \int \frac{Y'_{n,s} d\sigma'}{D} \right) = \sum A_{n,s} Y_{n,s}$$

or

$$\sum \frac{A_{n,s}}{T_{n,s}} Y_{n,s} \left(1 - \frac{\nu}{\nu_{n,s}} \right) = \sum A_{n,s} Y_{n,s}$$

This is an identity.

Thus, the general solution can be obtained if the terms corresponding to the conditions $T_{m,r} = 0$ are added to the particular solution (4.2.12). These terms are composed of the spherical functions $Y_{m,r}$. It is, then,

$$(4.2.13) \quad \xi = \sum \frac{A_{n,s}}{T_{n,s}} Y_{n,s} + \sum a_{m,r} Y_{m,r}$$

where $a_{m,r}$ are arbitrary constants. Their number does not exceed three. The expression (4.2.13) is a general solution. This is proved by the fact that a continuous function (ξ) is completely determined if the values of all integrals $\int \xi Y_{n,s} d\sigma$ are given. These integrals either are determined by the coefficients $A_{n,s}$ or can be found if we assume that the constants $a_{m,r}$ have some definite values. The function (4.2.12) has a finite value. It has been also proved by Liapounov that if \mathcal{E} is subjected to conditions similar to (4.1.9) for ζ , the function ξ will also satisfy the same conditions.

It will be shown in the next section under which conditions the fundamental equation in the theory of figures of equilibrium is reduced by Liapounov to a set of integral equations of the type (4.2.1) for heterogeneous fluids. A similar transformation has been used, however, in the earlier work of Liapounov, namely, in the theory of homogeneous figures of equilibrium which differ but little from an ellipsoid. On representing the coordinates of a point at the free surface by the formulae

$$\begin{aligned}x &= \sqrt{\varrho + \zeta + 1} \sin \vartheta \cos \psi \\y &= \sqrt{\varrho + \zeta + 1} \sin \vartheta \sin \psi \\z &= \sqrt{\varrho + \zeta} \cos \vartheta\end{aligned}$$

instead of (4.1.2) and assuming that ζ can be expanded in a series $\sum \zeta_i$ Liapounov obtained integral equations of the form (4.2.1) for the functions ζ_i . The integral equation of Liapounov (4.2.1) has been also investigated by Crudeli [1]. He could show that it can be reduced to a Fredholm equation of the second kind having a finite kernel.⁷

4.3. Transformation of the Fundamental Equation

By writting the equations of a level surface F_a in a parametrical form we shall have small values of the function $\zeta(a, \vartheta, \psi)$ if we assume that the stratification in a figure of equilibrium differs but little from the ellipsoidal one. In each case, however, the function can be expressed in terms of coordinates x, y, z , i.e., $\zeta(a, \vartheta, \psi) = \zeta^*(x, y, z)$.

The equation of level surfaces is, then, by (4.1.2)

$$(4.3.1) \quad \frac{x^2}{\varrho + 1} + \frac{y^2}{\varrho + q} + \frac{z^2}{\varrho} = a^2(1 + \zeta^*)^2$$

and the free surface will be given by

$$(4.3.2) \quad \frac{x^2}{\varrho + 1} + \frac{y^2}{\varrho + q} + \frac{z^2}{\varrho} = (1 + \bar{\zeta}^*)^2$$

where $\bar{\zeta}^*$ is the value of ζ at this surface and $a = 1$.

Since the density is a function of the parameter a and $\kappa = f(\rho)$, the pressure p can also be expressed in terms of this parameter, $p = p(a)$. Therefore, in the equation (1.1.5) the right-hand member is a function of a and the equation of level surfaces which follows from equations of motion of the liquid is

⁷ To our knowledge Crudeli is the only investigator who called the solutions of this equation Liapounov's figures of equilibrium.

$$(4.3.3) \quad U + \frac{\omega^2}{2f} (x^2 + y^2) = \text{funct. } (a)$$

The potential U can be expanded in series (4.1.15). Thus, the left-hand member of (4.3.3) becomes a function of ζ . This equation is now the fundamental equation from which the function ζ can be found. If the law of densities is given in the form (4.1.1) a transformation

$$(4.3.4) \quad U = \Psi + \delta\Phi$$

will introduce two new expressions Ψ and Φ which can be derived from U . To obtain these functions the factors 1 or $\varphi(a')$ must be taken in (4.1.7) instead of κ' . Then, if we expand the function Ψ or Φ in a series

$$(4.3.5) \quad \Psi = \sum_{i=0}^{\infty} \Psi_i \quad \Phi = \sum_{i=0}^{\infty} \Phi_i$$

the functions Ψ_i and Φ_i are determined by the formulae (4.3.16) where, obviously κ' must be put equal to 1 or to $\varphi(a')$. The first term Ψ_0 , represents the potential of a homogeneous ellipsoid E . Now, in the second equation (4.1.16), if $\kappa' = 1$ the integration with respect to a' yields

$$(4.3.6) \quad \Psi_1 = 2\zeta\Psi_0 + \Delta \int \frac{(\bar{\zeta}' - \zeta)}{D(a, 1)} d\sigma'$$

For further transformation we restrict ourselves by an ellipsoid of revolution. Let $M(x, y, z)$ be a point of an ellipsoid E_a . It is

$$(4.3.7) \quad \begin{aligned} x &= a\sqrt{\varrho + 1} \sin \vartheta \cos \psi & y &= a\sqrt{\varrho + 1} \sin \vartheta \sin \psi \\ z &= a\sqrt{\varrho} \cos \vartheta & \Delta &= (\varrho + 1)\sqrt{\varrho} \end{aligned}$$

It is easy to see that the variable l in (2.1.5) can be expressed in terms of ϱ if one takes the axes $a\sqrt{\varrho + 1}$ and $a\sqrt{\varrho}$ used in (4.3.7). It is $l = 1/\sqrt{\varrho}$. On writing now the constant in (2.2.1) in the form

$$2\pi f \Delta \tan^{-1} \frac{1}{\sqrt{\varrho}}$$

and using the equations (4.3.1) and (2.2.8) the function Ψ_0 is given by the equation

$$(4.3.8) \quad \Psi_0 = 2\pi\Delta \left[\tan^{-1} \frac{1}{\sqrt{\varrho}} - \frac{1}{2}a^2(\varrho+1)\sin^2\vartheta \left(\tan^{-1} \frac{1}{\sqrt{\varrho}} - \frac{\sqrt{\varrho}}{\varrho+1} \right) - a^2\varrho \cos^2\vartheta \left(\frac{1}{\sqrt{\varrho}} - \tan^{-1} \frac{1}{\sqrt{\varrho}} \right) \right]$$

We put now

$$(4.3.9) \quad C = \frac{1}{2} \int_{\varrho}^{\infty} \frac{dt}{(t+1)\sqrt{t}} = \tan^{-1} \frac{1}{\sqrt{\varrho}}$$

The integral

$$J = \int \frac{d\sigma'}{D(a, 1)}$$

represents the potential of a layer on the ellipsoid E at an interior point M . It is equal to its value at the center of E where $D(a, 1) = D(0, 1)$. By formula (4.1.12) we have in case of an ellipsoid

$$D(0, 1) = (\varrho + \sin^2\vartheta')^{\frac{1}{2}}$$

On putting $\cos\vartheta = \bar{\mu}$ we obtain

$$\begin{aligned} \int \frac{d\sigma'}{D(0, 1)} &= \int_0^{\pi} \int_0^{2\pi} \frac{\sin\vartheta' d\vartheta' d\varphi'}{\sqrt{\varrho + \sin^2\vartheta'}} \\ &= 2\pi \int_{-1}^1 \frac{d\bar{\mu}'}{\sqrt{\varrho + 1 - \bar{\mu}'^2}} = 4\pi \sin^{-1} \frac{1}{\sqrt{\varrho + 1}} \end{aligned}$$

Since

$$\sin^{-1} \frac{1}{\sqrt{\varrho + 1}} = \tan^{-1} \frac{1}{\sqrt{\varrho}}$$

and by (4.3.9) $J = 4\pi C$ it is

$$(4.3.10) \quad \Psi_1 = (2\Psi_0 - 4\pi C\Delta)\xi + \Delta \int \frac{\xi' d\sigma'}{D(a, 1)}$$

Because of (4.3.4) and (4.3.5) we can write the equation (4.3.3) in the form

$$(4.3.11) \quad \Psi_0 + \Psi_1 + \frac{\omega^2}{2f}(x^2 + y^2) = \text{funct.}(a) - \delta\Phi - \sum_{i=2}^{\infty} \Psi_i$$

Since E is a figure of equilibrium, the condition

$$\Psi_0 + \frac{\omega^2}{2f} (x_a^2 + y_a^2) = h_a$$

must be satisfied on each surface E_a , the sum in parentheses being the square of the distance of a point on E_a from the axis of rotation, and h_a is a number not varying on this surface. It depends however, on a . The expression for h_a can be determined from (4.3.8) by putting $\vartheta = 0$. It will be

$$h_a = 2\pi\Delta \left[\tan^{-1} \frac{1}{\sqrt{\varrho}} - a^2 \left(\sqrt{\varrho} - \varrho \tan^{-1} \frac{1}{\sqrt{\varrho}} \right) \right]$$

If we put now

$$(4.3.12) \quad R = \sqrt{\varrho} - \varrho \tan^{-1} \frac{1}{\sqrt{\varrho}}$$

we obtain

$$(4.3.13) \quad h_a = 2\pi\Delta (C - a^2 R)$$

By (4.3.7) it is

$$x_a^2 + y_a^2 = a^2(\varrho + 1) \sin^2 \vartheta$$

and the condition for the surface E_a takes the form

$$(4.3.14) \quad \Psi_0 + \frac{\omega^2}{2f} a^2(\varrho + 1) \sin^2 \vartheta = 2\pi\Delta (C - a^2 R)$$

The expression for Ψ_0 determined by this equation and Ψ_1 from (4.3.10) can be inserted into (4.3.11). Then, we have

$$\begin{aligned} 2\pi\Delta (C - a^2 R) - \frac{\omega^2}{2f} a^2(\varrho + 1) \sin^2 \vartheta - \left[4\pi\Delta a^2 R + \frac{\omega^2}{f} a^2(\varrho + 1) \sin^2 \vartheta \right] \zeta \\ + \frac{\omega^2}{2f} s^2 + \Delta \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = - \sum_{i=2}^{\infty} \Psi_i - \delta\Phi + \text{funct. } (a) \end{aligned}$$

where

$$s^2 = a^2(1 + \zeta)^2 (\varrho + 1) \sin^2 \vartheta$$

since the condition (4.3.3) holds for level surfaces F_a which differ from E_a . Rearranging the terms we obtain

$$\begin{aligned}
 (4.3.15) \quad & -4\pi\Delta a^2 R\zeta + \Delta \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = 2\pi\Delta (a^2 R - C) \\
 & - \frac{\omega^2}{2f} a^2 (\varrho + 1) \sin^2 \vartheta \zeta^2 - \sum_{i=2}^{\infty} \Psi_i - \delta\Phi + \text{funct. } (a)
 \end{aligned}$$

This equation is of the type

$$(4.3.16) \quad R\zeta - \frac{1}{4\pi a^2 \Delta} \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = W + P(a)$$

We can put, for $a \neq 0$,

$$\begin{aligned}
 (4.3.17) \quad W = & \frac{\omega^2}{8f\pi\Delta} (\varrho + 1) \sin^2 \vartheta \zeta^2 \\
 & + \frac{1}{4\pi a^2 \Delta} \left\{ \sum_{i=2}^{\infty} \Psi_i - \Psi_2(0) + [\Phi - \Phi(0)]\delta \right\}
 \end{aligned}$$

$$(4.3.18) \quad P(a) = -\frac{1}{4\pi a^2 \Delta} \{ \text{funct. } (a) + 2\pi\Delta (a^2 R - C) - \Psi_2(0) - \Phi(0)\delta \}$$

The terms $\Psi_2(0) = [\Psi_2]_{a=0}$ and $\Phi(0) = [\Phi]_{a=0}$ do not depend on ϑ and ψ . As to the function W it depends on ζ^2 and terms of a higher order.

On taking into account all expressions for W , Ψ_i , and Φ we can easily see that equation (4.3.16) belongs to the class of integro-differential equations. The fundamental equation (4.3.3) is now written in the form (4.3.16) for figures of equilibrium which satisfy the conditions mentioned above.

The solution of this equation ζ is a function of two variables a and ϑ since it is written for an ellipsoid of revolution, and of a parameter δ . The general case where $\zeta = \zeta(a, \vartheta, \psi)$ was also solved by Liapounov. There is a second unknown function in the equation (4.3.15). It is denoted by the symbol $\text{funct. } (a)$. Nevertheless it is possible to determine the function ζ since we have the second condition. This is the equation (4.1.6).

Now, to solve the problem we have, according to Liapounov, to assume that the function ζ is expanded in a series

$$(4.3.19) \quad \zeta = \sum_{i=1}^{\infty} \zeta_i \delta^i$$

where the term ζ_0 is omitted to eliminate the case of a homogeneous liquid. If we put

$$(4.3.20) \quad W = \sum_{i=1}^{\infty} W_i \delta^i$$

equation (4.3.16) yields the set of conditions

$$(4.3.21) \quad R\zeta_i - \frac{1}{4\pi a^2} \int \frac{\bar{\zeta}_1' d\sigma'}{D(a, 1)} = W_i + P_i$$

for the coefficients ζ_i .

Besides these equations, substituting (4.3.19) into (4.1.6) we obtain the conditions

$$(4.3.22) \quad \begin{aligned} \int \zeta_1 d\sigma &= 0 & \int \zeta_2 d\sigma &= - \int \zeta_1^2 d\sigma \\ \int \zeta_3 d\sigma &= -2 \int \zeta_1 \zeta_2 d\sigma - \frac{1}{3} \int \zeta_1^3 d\sigma, \dots, \int \zeta_i d\sigma = N_i \end{aligned}$$

where N_i depends on $\zeta_1, \zeta_2, \dots, \zeta_{i-1}$. In order to eliminate the functions P_i in (4.3.21) we multiply this equation by $d\sigma$ and integrate over the unit sphere Σ . Then,

$$(4.3.23) \quad RN_i - \frac{1}{4\pi a^2} \int \bar{\zeta}_i' d\sigma' \int \frac{d\sigma}{D(a, 1)} = \int W_i d\sigma + 4\pi P_i$$

In the surface integral J_1 , the denominator $D(a, 1)$ is the distance of a point on E_a to some point on E . If we put $\zeta = \zeta' = 0$ in (4.1.4), (4.1.5), (4.1.7), the last-named equation will represent the potential of a homogeneous layer between the surfaces E_a and E_{a+a} at a point of E . The density of this layer is equal to unity, and we obtain the expression

$$a^2 da \Delta(\varrho) \int \frac{d\sigma}{D(a, 1)} = a^2 \Delta(\varrho) da J_1$$

On changing the notations in the well-known expression for the potential of a homogeneous ellipsoid (see, for example, Appell [1] Vol. III) at a point $M(x, y, z)$ outside it we obtain this potential in the form

$$(4.3.24) \quad U_e = \pi \Delta(\varrho) \int_e^\infty \left(a^2 - \frac{x^2}{t+1} - \frac{y^2}{t+q} - \frac{z^2}{t} \right) \frac{dt}{\Delta(t)}$$

where ε is a positive root of the equation

$$(4.3.25) \quad \frac{x^2}{t+1} + \frac{y^2}{t+q} + \frac{z^2}{t} = a^2$$

which corresponds to the ellipsoid confocal to E_a and passing through M . Now, the potential of the layer just considered will be

$$DU_\varepsilon = 2\pi\Delta(\varrho)ada \int_\varepsilon^\infty \frac{dt}{\Delta(t)} - \pi\Delta(\varrho) \left(a^2 - \frac{x^2}{\varepsilon+1} - \frac{y^2}{\varepsilon+q} - \frac{z^2}{\varepsilon} \right) \frac{d\varepsilon}{\Delta(\varepsilon)}$$

since U_ε and ε are functions of the parameter a . The expression in brackets vanishes because of (4.3.25), and we obtain comparing two expressions for the potential of the layer

$$(4.3.26) \quad J_1 = \int \frac{d\sigma}{D(a, 1)} = \frac{2\pi}{a} \int_\varepsilon^\infty \frac{dt}{\Delta(t)}$$

For an ellipsoid of revolution ($q=1$), (4.3.26) can be expressed in terms of elementary functions. It is

$$(4.3.27) \quad J_1 = \frac{4\pi}{a} \tan^{-1} \frac{1}{\sqrt{\varepsilon}}$$

where ε is a positive root of the equation

$$(4.3.28) \quad \frac{\varrho+1}{t+1} \sin^2 \vartheta' + \frac{\varrho}{t} \cos^2 \vartheta' = a^2$$

This equation is readily obtained from (4.3.25) and (4.1.2) if those equations are written for the case $\zeta = 0$.

Eliminating P_i by (4.3.26), (4.3.23), and (4.3.21) we obtain

$$(4.3.29) \quad R\zeta_i - \frac{1}{4\pi a^2} \int \bar{\zeta}_i' \left(\frac{1}{D(a, 1)} - \frac{1}{2a} \int_\varepsilon^\infty \frac{dt}{\Delta(t)} \right) d\sigma' \\ = W_i - \frac{1}{4\pi} \int W_i d\sigma + \frac{1}{4\pi} R N_i$$

The right-hand member of this equation depends on functions $\zeta_1, \dots, \zeta_{i-1}$. If these functions are determined (4.3.29) can be used to find the next function ζ_i .

Let us put now in (4.3.29) $a = 1$ and denote by a bar the cor-

responding value of a function of this parameter. For $\bar{\zeta}_i$ we have, then, the equation

$$(4.3.30) \quad R\bar{\zeta}_i - \frac{1}{4\pi} \int \frac{\bar{\zeta}_i d\sigma'}{D(1, 1)} = \bar{W}_i + \text{const.}$$

This is an integral equation of the type (4.2.1), and we have just followed the method used by Liapounov to solve it. For the ellipsoids of Jacobi the expression W_i will depend on both angles, ϑ and ψ , but we will consider an ellipsoid of revolution only. The formula (4.3.17) shows that W_i is a function of ϑ . The last two terms in (4.3.29) yield constants. If now a function $\bar{\zeta}_i$ is determined from (4.3.30), the corresponding function ζ_i may be directly computed by (4.3.29). Thus, the solution of the integro-differential equation (4.3.16) is reduced to the solution of an infinite set of integral equations (4.3.30). The terms of the series (4.3.19) can be computed successively from this set. The effective computation of the first three terms in case of a heterogeneous liquid required, however, several hundreds of pages in the quoted memoir of Liapounov. It should be noted that in order to exclude any doubt about the existence of these figures of equilibrium, Liapounov gave a proof for every intermediate step. The important progress brought by this method consists of the possibility of using any desired approximation. On the contrary, as mentioned before, Poincaré's analysis was restricted to the first approximation only.⁸

We can give here only the final expressions of Liapounov for the first approximation, i.e., the function ζ_1 . It is like all ζ_i represented by a regular Laplace series

$$(4.3.31) \quad \zeta_1 = \sum_{n=1}^{\infty} S A_{n,i}^{(1)} Y_{2n,4l}$$

(Liapounov [9], p. 170). The symbol S denotes the summation with respect to the index l .

⁸ A simplified computation in case of an ellipsoid of revolution was given by Jardeitzky [3] for the problem of a zonal rotation.

The first step to obtain this result is to expand the function W_1 in a series

$$(4.3.32) \quad W_1 = \sum_{n=1}^{\infty} S_{n,l} A_{n,l} Y_{2n,4l} + \text{funct. } (a)$$

where

$$(4.3.33) \quad A_{n,l} = -\frac{2\pi}{\gamma_{2n,4l}} \begin{pmatrix} 2n, 2l \\ 0, 0 \end{pmatrix} \frac{\mathfrak{F}_{2n,4l}}{4n+1} \int_0^1 \varphi(av) (1-v^2) v^2 \frac{\Omega_{n,0}(v^2)}{\Omega_{n,0}(1)} dv$$

and by (4.2.11)

$$\gamma_{2n,4l} = \int (Y_{2n,4l})^2 d\sigma$$

The coefficients denoted by an array are by definition

$$(4.3.34) \quad \begin{pmatrix} 2n, 2l \\ 2i, 2j \end{pmatrix} = \frac{(4n+1)\Delta}{\gamma_{2i,4i} \mathfrak{E}_{2i,4i} \mathfrak{F}_{2n,4l}} \int \frac{Y_{2i,4i} Y_{2n,4l}}{H} d\sigma$$

where

$$(4.3.35) \quad \begin{aligned} \mathfrak{E}_{2n,4l}(u) &= E_{2n,4l}(i\sqrt{u}) \\ \mathfrak{F}_{2n,4l}(u) &= \frac{2n+1}{2} \mathfrak{E}_{2n,4l}(u) \int_u^{\infty} \frac{du}{\mathfrak{E}_{2n,4l}^2 \sqrt{u(u+1)(u+q)}} \end{aligned}$$

and

$$(4.3.36) \quad \begin{aligned} H &= (\varrho + \mu^2)(\varrho + \nu^2) = (\varrho + q)\varrho \sin^2 \vartheta \cos^2 \psi \\ &\quad + \varrho(\varrho + 1) \sin^2 \vartheta \sin^2 \psi + (\varrho + 1)(\varrho + q) \cos^2 \vartheta \end{aligned}$$

\mathfrak{E} and \mathfrak{F} are old notations for Lamé functions (see, for example, Heine, E., *Theorie der Kugelfunctionen*, 1878, Vol. I, p. 358, 385). It is $E(\mu) = M$, $E(\nu) = N$, etc. in notations used in Chapter III of this book.

Now, the function Ω used in (4.3.33) is a solution of the differential equation for a hypergeometric series. It is

$$(4.3.37) \quad \Omega_{m,n}(z) = \mathcal{F}(n+1-m, n+m+\frac{3}{2}, 2n+\frac{3}{2}, z)$$

Finally it is

$$(4.3.38) \quad A_{n,l}^{(1)} = \frac{\bar{A}_{n,l}}{T_{n,l}}$$

where $T_{n,l}$ depend on eigenvalues of the integral equation as given by (4.2.3).

These few formulae show the extreme complexity of Liapounov's theory. It was applied by Liapounov in his memoir to the case of a homogeneous liquid, and the existence of figures of equilibrium which differ but little from an ellipsoid was confirmed once more.

The last conclusion of the theory is as follows (Liapounov [9] p. 436): *in all cases* (determined by the conditions mentioned above) *the surface of a figure of equilibrium is determined by the equation of the form*

$$\frac{X^2}{\varrho + 1} + \frac{Y^2}{\varrho + q} + \frac{Z^2}{\varrho} = 1 + \sum_{i=1}^{\infty} \Phi_i(X, Z) \alpha^i \quad (A)$$

where Φ_i is an entire function of X, Z of the degree $(m-2)i + 2$. It is an even function of Z and an even or odd function of X when the product mi is even or odd respectively. As to α in this equation it is a parameter in terms of which the function ζ could be expressed. In the investigations concerning the homogeneous liquids, Liapounov made use of the expression $\alpha = \sqrt{\int \zeta^2 d\sigma}$ and also of the second parameter $\eta = (\omega^2 - \omega_0^2)/2f$.

The fact proved by Liapounov that Φ_i are all even in Z yields immediately the symmetry of every figure of equilibrium with respect to the equatorial plane. The general form of this statement was suggested by Lichtenstein (Chapter V).

The importance of the last result of Liapounov's theory is evident. Equation (A) represents the most general known solution of the problem on figures of equilibrium.

4.4. Clairaut's Equation and More General Equations

The formulae quoted in the preceding section can be easily transformed for the case of spheroids. On omitting the factors $\sqrt{\varrho + 1}$, $\sqrt{\varrho + q}$, $\sqrt{\varrho}$ in (4.1.2) we obtain level surfaces which differ but little from spheres having radii equal to a . We shall denote the radius of the free surface of an undisturbed heterogeneous mass by A . Let $\kappa = \kappa(a)$ be the density and ω the angular velocity of rotation having a small value. The formulae used by Liapounov (1903, 1904) to derive the equation of Clairaut

are particular forms of (4.1.7), (4.1.15), (4.1.16), and (4.3.3). There is no more small parameter δ in the law of densities, and the function ζ representing the departure of a level surface from a sphere was expanded by Liapounov in the series

$$(4.4.1) \quad \zeta = \zeta_1 \sigma + \zeta_2 \sigma^2 + \dots$$

where

$$(4.4.2) \quad \sigma = \frac{\omega^2 A^2}{f m}$$

It is evident that this is the ratio

$$\frac{v_E^2}{A} : \frac{f m}{A}$$

where $m = 4\pi \int_0^A \kappa a^2 da$ is the total mass. A set of equations for coefficients ζ_i resulting from (4.3.30) can be resolved if we assume that

$$\zeta_i = \sum_{j=0}^{\infty} \zeta_{ij} P_j(\bar{\mu})$$

where P_j are Legendre's polynomials and $\bar{\mu} = \cos \vartheta$. The functions ζ_{ij} must satisfy, then, Legendre-Laplace equations

$$(4.4.3) \quad \zeta_{ij} \int_0^a \kappa a^2 da - \frac{a^{-j}}{2j+1} \int_0^a \kappa d(a^{j+3} \zeta_{ij}) - \frac{a^{j+1}}{2j+1} \int_a^A \kappa d(a^{2-j} \zeta_{ij}) = a^3 W_j(a)$$

Clairaut's equation (Section 2.4) is obtained for $i = 1$, $j = 2$, if we put

$$(4.4.4) \quad \zeta_{12} = -\frac{8\pi}{15} \frac{\alpha_a}{\sigma}, \quad W_2 = -\frac{m}{15A^3}$$

where α_a is ellipticity of the a -layer. From (4.4.3) the ellipticity can be determined in terms of a . On differentiating (4.4.3) twice, the Clairaut differential equation is readily obtained. To solve these equations a law of densities must obviously be given.

The solution of Clairaut's problem is now represented by

$$(4.4.5) \quad \zeta = \zeta_1 \sigma = \frac{5\sigma}{4\pi} \zeta_{12} P_2(\bar{\mu})$$

In order to extend this theory to those cases where the density is not a continuous function of the parameter a , Liapounov considered also equations of the form

$$(4.4.6) \quad \tilde{\zeta} \int_0^a \kappa a^2 da - \frac{a^{-j}}{2j+1} \int_0^a \kappa \Delta(a^{j+3} \tilde{\zeta}) - \frac{a^{j+1}}{2j+1} \int_a^A \kappa \Delta(a^{2-j} \tilde{\zeta}) = a^3 W(a)$$

for all values of a in the interval $(0, A)$. The symbol S in this equation denotes a Stiltjes integral, and, therefore, (4.4.6) is an extension of the known integro-differential equations in which the Riemann integrals are involved.

Lichtenstein's Investigations

5.1. Homogeneous Mass

The complexity and extreme length of Liapounov's theory are due to the difficulties of the problem of equilibrium of a liquid mass and, of course, to the desire not to leave out the proof of any point of the theory. A modification of the method of Liapounov has been given by Lichtenstein [2] in a series of papers. His work: *Gleichgewichtsfiguren rotierender Flüssigkeiten* [1] represents the synthesis of these investigations. Having discovered an exact method for the solution of a functional equation (4.3.3) which reduces it to an integro-differential one as well as to a set of integral equations in the special case considered in Chap. IV, Liapounov did not present his method in a more general form. He did not even make use of these names for the types of equations just mentioned. A generalization has been found by Lichtenstein. Adding some new statements, Lichtenstein was able to discuss the general problem of figures of equilibrium in the vicinity of a known one. Poincaré postulated the existence of new figures of equilibrium which differ but little from any given figure of equilibrium of a homogeneous liquid mass. Making no assumption about the shape of this primary figure, Lichtenstein derived the fundamental integro-differential equation for such a general problem, and this equation is of the type considered by Liapounov [1] (Chap. IV). The method of successive approximations and Fredholm's theory provided the base for the solution of this equation. All these generalizations, however, concern in the first place the mathematical treatment of the problem, and no new types of figures of equilibrium are actually computed.

According to Lichtenstein, it is convenient to make use of Gauss*

parameters ξ and η and to write the equation of the free surface of a homogeneous liquid mass in the form

$$(5.1.1) \quad x = F(\xi, \eta), \quad y = G(\xi, \eta), \quad z = H(\xi, \eta)$$

This surface is not restricted to be a single boundary of a continuous mass. It can have several separated parts, but it is assumed that

$$(5.1.2) \quad \left[\frac{\partial(F, G)}{\partial(\xi, \eta)} \right]^2 + \left[\frac{\partial(G, H)}{\partial(\xi, \eta)} \right]^2 + \left[\frac{\partial(H, F)}{\partial(\xi, \eta)} \right]^2 \neq 0$$

i.e., there are no singular points on it.

Let S now be the surface of a figure of equilibrium of a homogeneous liquid mass corresponding to the angular velocity ω . We assume that for a new value of angular velocity ω_1 , differing but little from ω we obtain in the neighborhood of S another surface S_1 . The equation of the new free surface of liquid can be written in the form

$$(5.1.3) \quad \zeta = \zeta(\xi, \eta, \omega_1)$$

where ζ is a small distance of a point on S_1 from the surface S measured along the normal to the latter. We assume that $|\zeta| < \varepsilon_0$, where ε_0 is a certain small number. The equations of the surface S_1 are

$$(5.1.1') \quad x = F_1(\xi, \eta), \quad y = G_1(\xi, \eta), \quad z = H_1(\xi, \eta)$$

and, if we write conditions of equilibrium (1.1.5) for S and S_1 and subtract, we obtain the equation of the form

$$(5.1.4) \quad \begin{aligned} U_1(F_1, G_1, H_1) - U(F, G, H) \\ = \frac{\omega^2}{2f} (F^2 + G^2) - \frac{\omega_1^2}{2f} (F_1^2 + G_1^2) + C' \end{aligned}$$

If the pressure at the boundary surface is zero and the volume of the liquid does not vary, the term C' vanishes. Lichtenstein assumes that now:

$$|\zeta|, \left| \frac{\partial \zeta}{\partial \xi} \right|, \left| \frac{\partial \zeta}{\partial \eta} \right| \leq \varepsilon^* < \varepsilon_0$$

These conditions are equivalent to Liapounov's inequalities (4.1.9).⁹ The fact that the latter hold has been proved by Liapounov, as mentioned above. If ξ, η, ζ^* are the values of parameters corresponding to any given point $P(x, y, z)$, the potential U and U_1 respectively can be expressed in terms of these parameters and we have

$$(5.1.5) \quad U(x, y, z) = W(\xi, \eta, \zeta^*), \quad U_1(x, y, z) = W_1(\xi, \eta, \zeta^*)$$

Then, a unit mass on S will be subjected to the force of attraction

$$(5.1.6) \quad f \frac{\partial}{\partial n} W(\xi, \eta, 0)$$

and the gravity acceleration will be

$$g = \frac{\partial}{\partial n} \left(fU + \frac{\omega^2 s^2}{2} \right)$$

If $\gamma = ds/dn$ is the cosine of the angle which the perpendicular to the axis of rotation makes with the normal to S , we obtain by (5.1.5)

$$(5.1.7) \quad g = f \frac{\partial}{\partial n} W(\xi, \eta, 0) + \omega^2 s \gamma = f \hat{g}$$

Let D denote the distance between two points: (ξ, η) and (ξ', η') . According to Lichtenstein we can assume that the left-hand member in (5.1.4) can be expanded in a series

$$(5.1.8) \quad f(U_1 - U) = U^{(1)} + U^{(2)} + \dots, \quad U^{(n)} = f \int_S \frac{1}{D^n} K^{(n)} d\sigma'$$

where $K^{(n)}$ is a polynomial of the n th degree in $\zeta, \zeta', \partial\zeta'/\partial\xi', \partial\zeta'/\partial\eta'$. Then,

$$(5.1.9) \quad \begin{aligned} f \frac{\partial}{\partial \xi} (U_1 - U) &= f \sum_{n=1}^{\infty} \int_S \frac{\partial}{\partial \xi} \left(\frac{1}{D^n} K^{(n)} \right) d\sigma' \\ f \frac{\partial}{\partial \eta} (U_1 - U) &= f \sum_{n=1}^{\infty} \int_S \frac{\partial}{\partial \eta} \left(\frac{1}{D^n} K^{(n)} \right) d\sigma' \end{aligned}$$

⁹ It is not even necessary to postulate the existence of $\partial\zeta/\partial\xi, \partial\zeta/\partial\eta$ according to the last investigation of Lichtenstein [4]. When ζ is determined it can be shown that these derivatives exist.

are proved to be convergent series as well as (5.1.8) even for complex values of ζ . It can also be shown that if κ is the density

$$(5.1.10) \quad U^{(1)} = f\zeta \frac{\partial}{\partial n} W(\xi, \eta, 0) + f \int_s \frac{\kappa'}{D} \zeta' d\sigma'$$

If we now put¹⁰

$$(5.1.11) \quad \omega_1^2 - \omega^2 = 2f\lambda, \quad F_1 = F + a\zeta, \quad G_1 = G + b\zeta, \quad H_1 = H + c\zeta$$

and insert the expressions (5.1.7), (5.1.8), (5.1.10), (5.1.11) in (5.1.4) we obtain the equation of the form

$$(5.1.12) \quad g\zeta + \int_s \frac{\kappa'}{D} \zeta' d\sigma' = C' - s^2\lambda + \Psi(\lambda, \zeta)$$

Since $aF + bG = s\gamma$ the function

$$(5.1.13) \quad \begin{aligned} \Psi(\lambda, \zeta) = & -\frac{\omega^2}{2f} (a^2 + b^2)\zeta^2 - 2s\gamma\lambda\zeta - (a^2 + b^2)\lambda\zeta^2 \\ & - \frac{1}{f} (U^{(2)} + U^{(3)} + \dots) \end{aligned}$$

It represents the sum of terms of the second and higher order with respect to the variables ζ and λ , while the coefficients g and κ'/D of ζ and ζ' are finite. Because of (5.1.8) the equation (5.1.12) is an integro-differential one and similar to (4.3.16) which has been derived by Liapounov for a particular kind of figures of equilibrium namely, for ellipsoids. If we take a certain definite value of ζ , there is a set of values of the parameters $\vartheta, \psi, e\zeta$ which under condition $0 \leq \varepsilon \leq 1$ corresponds to a set of bodies considered by Liapounov. To $\xi, \eta, \delta\zeta$, when $0 \leq \delta \leq 1$, it corresponds to a set of bodies between a given figure of equilibrium and a new one in Lichtenstein's theory.

In the equation (5.1.12) g has a negative value. As in Liapounov's theory, the solution of (5.1.12) is based on the discussion of the homogeneous linear integral equation

¹⁰ This parameter λ is identical to η introduced by Liapounov and mentioned at the end of Section 4.3.

$$(5.1.14) \quad \hat{g}\zeta + \int_S \frac{\kappa'}{D} \zeta' d\sigma' = 0$$

This can be easily transformed into an integral equation with a symmetric kernel by the substitution $u = \zeta \sqrt{-\hat{g}h}$. There are two trivial solutions of this equation. They correspond to an evident property of each figure of equilibrium. When it is transferred, namely, along the axis of rotation or it is rotated by some angle with respect to the axis, it remains a figure of equilibrium corresponding to the same values of potential and angular velocity. These two solutions are $u_1 = \text{const.}$ and $u_2 = \text{const.}$ ($Fb - Ga$) and correspond to the conditions $T_{1,0} = 0$, $T_{2,3} = 0$ of Liapounov's theory. (See (4.2.4).)

As is well known from the theory of linear-integral equations, the eigenfunctions (u_k) of the transformed equation (5.1.14) can be normalized and orthogonalized. Then, it is

$$(5.1.15) \quad \int_S \kappa \hat{g} u_j u_l d\sigma = 0, \text{ for } j \neq l, \quad \int_S \kappa \hat{g} u_j^2 d\sigma = -1$$

If ϑ and ψ are spherical polar coordinates and we have a figure of rotation, the eigenfunctions are either of the type $F(\vartheta)$ or $F(\vartheta) \cos k\psi$ or $F(\vartheta) \sin k\psi$.

If we put $D(P, P_1)$ for distance between points P and P_1 and

$$(5.1.16) \quad \frac{1}{D} = N(P, P_1) + \sum_{i=1}^m \hat{g} \hat{g}' u_i u_i'$$

the equation

$$(5.1.17) \quad \hat{g}\zeta + \int_S \kappa' N \zeta' d\sigma' = 0$$

has no solutions representing eigenfunctions as it has been proved by the theory of integral equations.

On the other hand (5.1.12) can be written in the form

$$(5.1.18) \quad \hat{g}\zeta + \int_S \kappa' N \zeta' d\sigma' = C' - s^2 \lambda + \Psi(\lambda, \zeta) + \sum_{i=1}^m \hat{g} M_i u_i$$

where

$$(5.1.19) \quad M_i = - \int_S \kappa' \hat{g}' u_i' \zeta' d\sigma'$$

Lichtenstein proved that this equation has a simple solution if λ , C' , M_1, \dots, M_m have sufficiently small values. This solution as well as its first derivatives with respect to the parameters ξ and η are continuous functions and satisfy the conditions (5.1.4). It can be evaluated by successive approximations. Let $Q(P, P_1)$ be the resolvent. Then (5.1.18) takes the form

$$(5.1.20) \quad \begin{aligned} \xi = & \frac{1}{\xi} (C' - s^2 \lambda) + \sum_{i=1}^m M_i u_i + \frac{1}{\xi} \Psi(\lambda, \xi) \\ & - \int_s Q(P, P_1) \left[\frac{1}{\xi'} (C' - s'^2 \lambda) + \sum_{i=1}^m M_i u'_i + \frac{1}{\xi'} \Psi(\lambda, \xi') \right] d\sigma' \end{aligned}$$

Since (see, for example, Courant and Hilbert [1])

$$(5.1.21) \quad \int_s Q(P, P_1) u'_i d\sigma' = 0 \quad i = 1, 2, \dots, m$$

on writing

$$(5.1.22) \quad C' - s^2 \lambda + \Psi(\lambda, \xi) = \Psi(C', \lambda, \xi)$$

(5.1.20) takes the form

$$(5.1.23) \quad \xi = \frac{1}{\xi} \Psi(C', \lambda, \xi) + \sum_{i=1}^m M_i u_i - \int_s Q(P, P_1) \frac{1}{\xi'} \Psi(C', \lambda, \xi') d\sigma'$$

Now, as in Liapounov's theory for a special kind of figures of equilibrium (Section (4.3)), the successive approximations are obtained from a set of integral equations. The first two of these equations are obtained from (5.1.18) by writing

$$(5.1.24) \quad \begin{aligned} \xi \zeta_1 + \int_s \kappa' N \zeta'_1 d\sigma' &= C' - s^2 \lambda + \sum_{i=1}^m \xi M_i u_i \\ \xi \zeta_2 + \int_s \kappa' N \zeta'_2 d\sigma' &= C' - s^2 \lambda + \sum_{i=1}^m \xi M_i u_i + \Psi(\lambda, \xi_1) \\ &\dots \dots \dots \end{aligned}$$

the second and other equations follow by an obvious transformation.

The equations (5.1.24) are nonhomogeneous linear-integral equations of the form

$$(5.1.25) \quad \dot{g}\zeta^* + \int_S \kappa' N \zeta^{*'} d\sigma' = C' - s^2 \lambda + \sum_{l=1}^m \dot{g} M_l u_l + \Psi(\lambda, \bar{\zeta}^*)$$

where $\bar{\zeta}^*$ is a given function.

According to the general theory of equations of this type, the solution of (5.1.25) is of the form

$$(5.1.26) \quad \zeta^* = P + \Theta(\lambda, \bar{\zeta}^*)$$

and in case of (5.1.24) we can put

$$(5.1.27) \quad P = \frac{1}{\dot{g}} (C' - s^2 \lambda) + \sum_{l=1}^m M_l u_l - \int_S Q(P, P_1) \frac{1}{\dot{g}'} (C' - s'^2 \lambda) d\sigma'$$

On taking into account (5.1.20), (5.1.21) we can easily see that by (5.1.26)–(5.1.27) the solutions of (5.1.24) are given as follows:

$$(5.1.28) \quad \begin{aligned} \zeta_1 &= P \\ \zeta_2 &= P + \Theta(\lambda, \zeta_1) \\ \zeta_3 &= P + \Theta(\lambda, \zeta_2) \\ &\vdots \end{aligned}$$

where Θ is the sum of second order and higher terms. It is by (5.1.18)

$$(5.1.29) \quad \dot{g}\Theta = - \int_S \kappa' N' \Theta' d\sigma' + \Psi(\lambda, \zeta)$$

The equation (5.1.18) is equivalent to the fundamental equation of the problem (5.1.12) if the conditions (5.1.19) are satisfied. On writing these conditions in the form

$$(5.1.30) \quad M_l + \int_S \kappa' \dot{g}' u_l' \zeta' d\sigma' = 0, \quad l = 1, 2, \dots, m$$

it can be easily seen that $M_1 = M_2 = 0$, because of the vanishing of u_1 and u_2 . For other values of subscripts these conditions are obviously analogous to the conditions (4.2.4) of Liapounov's theory, i.e., correspond in general to some figures of bifurcation. Thus, a new set of figures of equilibrium can be expected in the vicinity of a given figure if this figure of a homogeneous liquid mass belongs to a special set determined by the conditions of bifurcation.

5.2. *Heterogeneous Mass*

In Chapter IV of his work, Lichtenstein considered some new heterogeneous figures of equilibrium in the vicinity of a given figure which is not necessarily homogeneous. These investigations have a general character and have never been carried out to some actual evaluations of new figures like those determined by Liapounov's theory. The density law used by Lichtenstein is of the form $\kappa = h(\alpha) + \delta\chi(\alpha)$ i.e., similar to (4.1.1). It is also assumed that the parameter δ takes small values, i.e., this is the case of the so-called "weakly inhomogeneous" figures. When the first term of the density is a certain function of a parameter α the primary figure of equilibrium, i.e., that in the vicinity of which new figures are sought, will be a heterogeneous one. Moreover, the density must not be continuous in the whole liquid. If there are some surfaces of discontinuity, Stieltjes' integrals instead of the Riemann integral must be used. As mentioned above this has been done by Liapounov in the generalization of Clairaut's problem.

Now, we have to show the principal steps in the derivation of the fundamental equation of the theory.

Let us assume that all surfaces of equal density intersect the z -axis. Then, the distance q of the point of intersection from the origing of coordinates can be taken as a parameter which will determine these surfaces S_q if they are not ring-shaped. The equation of S_q can now be written in the form

$$(5.2.1) \quad x = x(\xi, \eta, q), \quad y = y(\xi, \eta, q), \quad z = z(\xi, \eta, q)$$

If V is the volume of a certain initial figure of equilibrium and its density is represented by $\kappa = h(\alpha)$, we have the condition

$$(5.2.2) \quad \oint_V \frac{h'}{D} dV' + \frac{\omega^2 s^2}{2} = C(q) = \text{const.}$$

to be satisfied on S_q . If the law of densities is changed to $\kappa = h(\alpha) + \delta\chi(\alpha)$ for the new set of surfaces of equal densities $S_{1\alpha}$ we shall have

$$(5.2.3) \quad x_1 = x + a\xi, \quad y_1 = y + b\xi, \quad z_1 = z + c\xi$$

and, since the angular velocity has also been changed,

$$(5.2.4) \quad f \int_{v_1} \frac{h' + \delta \chi'}{D_1} dV'_1 + \frac{\omega_1^2 s_1^2}{2} = \text{const.} = C_1(q)$$

Hence

$$(5.2.5) \quad f \int_{v_1} \frac{h'}{D_1} dV'_1 - f \int_v \frac{h'}{D} dV' + f \delta \int_{v_1} \frac{\chi'}{D_1} dV'_1 + \frac{\omega_1^2 s_1^2}{2} - \frac{\omega^2 s^2}{2} = C_1 - C = fC'(q)$$

It was shown by Lichtenstein that the difference represented by the first two terms of the left-hand member can be expanded in a uniformly convergent series.

$$(5.2.6) \quad W_1 - W = W^{(1)} + W^{(2)} + \dots$$

The properties of this series are quite similar to those of the series (5.1.8)–(5.1.9). It has been obviously assumed that new surfaces of equal density differ but little from the initial ones and that, of course, the difference $\omega_1 - \omega$ is small enough. Because of the variability of h , equation (5.1.10) must be replaced by a more general equation as follows:

$$(5.2.7) \quad W^{(1)} = \zeta \frac{\partial W}{\partial n} + f \int_{S_q} \frac{h'}{D} \zeta' d\sigma' - f \int_v \zeta' \frac{\partial h'}{\partial n'} \frac{1}{D} dV'$$

This expression holds for a continuous h . However, if there is a discontinuity of density along a surface S_* we have to put $[h'] = h(q^* + 0) - h(q^* - 0)$ and

$$(5.2.8) \quad W_*^{(1)} = W^{(1)} - f \int_{S_*} [h'] \frac{\zeta'}{D} d\sigma'$$

By (5.2.5)–(5.2.8) we obtain

$$(5.2.9) \quad \zeta \frac{\partial W}{\partial n} + f \int_s \frac{h'}{D} \zeta' d\sigma' - f \int_v \zeta' \frac{\partial h'}{\partial n'} \frac{1}{D} dV' = fC' - W^{(2)} - W^{(3)} - f \delta \int_{v_1} \frac{\chi'}{D_1} dV'_1 - f s^2 \lambda - \frac{\omega_1^2}{2} \zeta^2 (a^2 + b^2) - (\omega^2 + 2f\lambda)(ax + by)$$

On putting again

$$\frac{\partial W}{\partial n} + \omega^2(ax + by) = f\bar{g}$$

the fundamental equation of the problem takes the form

$$\begin{aligned} (5.2.10) \quad & \bar{g}\bar{\zeta} + \int_s \frac{h'}{D} \bar{\zeta}' d\sigma' - \int_{s_*} [h'] \frac{\bar{\zeta}'}{D} d\sigma' - \int_v \bar{\zeta}' \frac{\partial h'}{\partial n'} \frac{1}{D} dV' \\ & = C' - \delta \int_{v_1} \frac{\chi'}{D_1} dV'_1 - \lambda s^2 - \frac{\omega_1}{2f} \bar{\zeta}^2 (a^2 + b^2) - 2\lambda \bar{\zeta} (ax + by) \\ & \quad - \frac{1}{f} (W^{(2)} + W^{(3)} + \dots) \end{aligned}$$

In the general discussion of this equation given by Lichtenstein, there are no points which in principle would differ from those given for the case of a homogeneous liquid. In order to obtain the equation (5.1.12) which holds for this simple case, we have to put in (5.2.10)

$$\frac{\partial h'}{\partial n'} = 0, \quad \chi = 0, \quad [h'] = 0$$

The equation of Clairaut is obtained in this discussion by putting $\omega = 0 (\lambda \neq 0)$.¹¹ Lichtenstein's suggestion is to take four parameters: δ , λ , $C'(\bar{q})$, and M_3 , where \bar{q} is the value of q corresponding to the free surface of liquid, and to assume that these parameters take small values, i.e., that their moduli are smaller than a certain given small number. Then, for $\bar{\zeta}$, its expansion in a power series

$$(5.2.11) \quad \bar{\zeta} = \sum a_{\nu_1 \nu_2 \nu_3 \nu_4} \lambda^{\nu_1} C'^{\nu_2} \delta^{\nu_3} M_3^{\nu_4}$$

can be used in order to compute new figures of equilibrium.

5.3. Some Other Results of Lichtenstein's Theory

The general theory outlined in preceding sections could be obviously applied to a set of problems that present a certain interest from the viewpoint of Celestial Mechanics. Lichtenstein

¹¹ i.e., starting with a figure of absolute equilibrium.

discussed, therefore, the figure of a liquid mass rotating in the field created by some external masses, Laplace's problem of the shape of a liquid surrounding a rigid spherical core, as well as the liquid cylinder and the exact shape of the ocean.

The basic point in which the treatment of all these problems differ is the shape of the given surface S in equation (5.1.14). In Liapounov's theory, the integral involved in this equation has been taken over the unit sphere and, therefore, the eigenfunctions could be expressed in terms of spherical, or Lamé functions. In case of some other surface S , the main difficulty will probably consist in the kind of functions by means of which the eigenfunctions can be found in the form suitable for further discussion of the problem.

A new extension of Liapounov's theory has been given by Lichtenstein [1] (Chap. V) in the application of the former's method to problems of this type: Suppose a given figure of a liquid mass satisfies conditions of equilibrium in first approximation; an exact figure of equilibrium in the vicinity of the first figure is to be found.

It is obvious that the difference of potentials of two such bodies can be expanded in a series similar to (5.2.6), since for this part of analysis no condition of equilibrium is to be used. Then, a function ζ can be defined to show departure in the shape of the second body from the first. Now the second figure satisfies the exact conditions of equilibrium while for the first only the first approximation holds. Let ω be the angular velocity of the figure of equilibrium. Assume $\omega(\delta)$ is a certain function of a parameter δ , and consider a set of figures of equilibrium corresponding to it. Then, in the way shown in Section (5.1) an equation analogous to (5.1.12) can be obtained with three parameters C' , λ , and δ .

Obviously attempts can be made to solve problems of this kind for values of C' , λ , and δ which are small enough. An essential difference between this application of the method and the theory given in preceding sections must be kept in mind. The equation like (5.1.12) is obtained by subtracting two exact equations. As for the new case, one of these conditions holds only in the first approximation as mentioned above.

The problems in which this approach could be of interest are according to Lichtenstein as follows: (a) a ring-shaped figure of equilibrium, (b) a ring with a central body, (c) a revolution of an almost spherical body around a mass point (central body); as a primary figure an ellipsoid of Roche could be taken, (d) in particular cases of the problem of n bodies when they are rotating as a rigid system about a common mass center and all have almost spherical shapes, (e) double- or multiple-star systems with a central body, (f) a system of concentric rings. Investigations of several students of Lichtenstein concerning these problems and related questions are listed in the Bibliography.

5.4. Symmetry of Figures of Equilibrium

All figures of equilibrium, the existence of which has been demonstrated, namely, the ellipsoids of Maclaurin and Jacobi, the Poincaré-Liapounov figures possess at least one plane of symmetry. This plane is perpendicular to the axis of rotation. The existence of it is a general property of figures of equilibrium and this fact has been proved in its general form for the first time by Lichtenstein [2].

Method of Wavre

6.1. Fundamental Equations

The difficulties concerning the series expansions of the factor $1/D$ in the expression for potential represented by the formulae (1.4.13) as well as the desideratum of Tisserand [1] (Vol. II, p. 317) were mentioned before. A method to overcome these difficulties has been suggested by Wavre and has been called the "Uniform Method." In order to obtain the fundamental equations the solution of which, by this method, yields the figures of equilibrium we write first the equation (1.1.5) in the form

$$(6.1.1) \quad \Phi(\kappa) = fU + Q$$

The right-hand member of (1.1.5) obviously depends only on the density κ if $\kappa = \varphi(\phi)$. For Q we shall have a more general expression in the next chapter, but in the case of figures of equilibrium Q is a linear function of s^2 and, therefore, $\nabla^2 Q = 2\omega^2$. We assume that Φ and its differential coefficients of the first two orders are continuous in the space V filled in with the fluid and that Φ is an analytical function in the part of space exterior to V . On taking into account the Poisson equation for the potential U , it follows from (6.1.1)

$$(6.1.2) \quad \nabla^2 \Phi = -4\pi f\kappa + 2\omega^2$$

We assume now that Φ at the boundary S has a constant value Φ_S . To find the expression for the value Φ_M at some point M , we have to make use of Green's formula in two different cases. Let M be an exterior point and $D = MM'$ its distance from a point M' of the body. Then, we have

$$\int_V \left(\frac{1}{D} \nabla^2 \Phi - \Phi \nabla^2 \frac{1}{D} \right) dV + \int_S \left(\frac{1}{D} \frac{d\Phi}{dn} - \Phi \frac{d}{dn} \frac{1}{D} \right) dS = 0$$

taking the inner normal. Since $\nabla^2(1/D) = 0$ and in the last term it is $\Phi = \Phi_s = \text{const.}$, Gauss' integral is equal to zero and the equation takes the form

$$(6.1.3) \quad \int_V \frac{1}{D} \nabla^2 \Phi dV + \int_S \frac{1}{D} \frac{d\Phi}{dn} dS = 0$$

It holds for the outer part of space and along the boundary surface. If M is an interior point and σ an arbitrary small sphere with the center at P , the surface integral in Green's formula is taken over S and σ . It will yield two terms, Gauss' integral for each of them being equal to 4π . Thus, we obtain

$$(6.1.4) \quad \int_V \frac{1}{D} \nabla^2 \Phi dV + \int_S \frac{1}{D} \frac{d\Phi}{dn} dS = 4\pi(\Phi_s - \Phi_M)$$

and this equation holds in the interior and on the boundary. Substituting (6.1.2) in the first term of (6.1.3) or (6.1.4) and taking into account, that

$$U = \int_V \frac{\kappa dV}{D}$$

we obtain

$$\int_V \frac{1}{D} \nabla^2 \Phi dV = -4\pi fU + 2\omega^2 \int_V \frac{dV}{D}$$

By (6.1.1) we have $\Phi_M = fU_M + Q_M$ and, therefore, (6.1.3) and (6.1.4) can be written in a symmetrical form

$$(6.1.5) \quad \frac{\omega^2}{2\pi} \int_V \frac{dV}{D} + \frac{1}{4\pi} \int_S \frac{1}{D} \frac{d\Phi}{dn} dS + Q_P = \begin{cases} \Phi_M \\ \Phi_S \end{cases}$$

On taking any level surface S_* , the potential U can be represented by the sum

$$(6.1.6) \quad U = U_1 + U_2 = \int_{V_*} \frac{\kappa dV}{D} + \int_L \frac{\kappa dV}{D}$$

where V_* is the volume bounded by S_* and L is the space between S and S_* . It is evident that if S_* is an external surface the density κ in the layer L must be taken equal to zero. On applying the formula (6.1.5) to the first term in (6.1.6) we obtain the equation

$$(6.1.7) \quad f \int_L \frac{\kappa}{D} dV + \frac{\omega^2}{2\pi} \int_{V_*} \frac{dV}{D} + \frac{1}{4\pi} \int_{S_*} \frac{1}{D} \frac{d\Phi}{dn} dS + Q_M = \begin{cases} \Phi_M \\ \Phi_{S_*} \end{cases}$$

The necessary and sufficient condition of equilibrium has been presented by Wavre in this form. As to the density distribution, it has been assumed that the density in a figure of equilibrium does not decrease toward the center. We have discussed this assumption in Section (1.3). The second equation (6.1.7) is a condition which must be satisfied on each level surface S_* in the interior of a rotating liquid mass, and its solution will determine the stratification in the mass interior as well as the free surface that is the figure of equilibrium. In order to solve this equation, Wavre suggested its transformation as follows. Since $d\Phi/dn = g$, we can write the second equation (6.1.7) in the form

$$(6.1.8) \quad \frac{1}{4\pi} \int_{S_*} \frac{g}{D} dS + f \int_L \frac{\kappa}{D} dV + \frac{\omega^2}{2\pi} \int_{V_*} \frac{dV}{D} + Q_M - \Phi_{S_*} = 0$$

The first term can now be interpreted as the potential of a simple layer on the boundary S_* of a "cavity" having the density equal

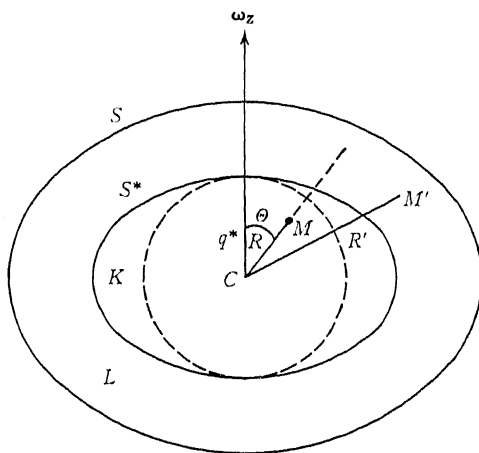


Figure 6

to g . In the interior of S_* all the terms in (6.1.8) are analytic functions. Therefore, the left-hand member of (6.1.8) will be

determined by its expansion in a Taylor series, for example, for its value in the vicinity of the center. To obtain this expression, it is assumed that the volume V_* of the cavity is divided in two parts. The third term in (6.1.8) can then be represented by a sum. At first we take the integral over the sphere of a radius q_* equal to the distance of the point of intersection of the surface and the axis of rotation. It remains the integral over the space K enclosed between this sphere and the surface S_* . If R is the distance of the point P in the "cavity," and θ the angle $(\mathbf{R}, \boldsymbol{\omega})$ we have

$$(6.1.9) \quad \int_{V_*} \frac{dV}{D} = 2\pi \left[q_*^2 - \frac{R^2}{3} \right] + \int_K \frac{1}{D} dV$$

Now, if the point M is close enough to C for a given S_* (in general an arbitrary one), one can use the second Laplace expansion of $1/D$ in a series (1.4.13). In fact, in

$$\frac{1}{D} = \frac{1}{R'} \sum_0^{\infty} \left(\frac{R}{R'} \right)^n P_n(\bar{\mu})$$

for $R < R'$ the latter condition holds for all three integrals in (6.1.8) the expression (6.1.9) being inserted in (6.1.8). In these equations R' is the distance of any element of integrals from C and $\bar{\mu} = \cos M'CM$. If we now take the conditions that coefficients of all R^n in (6.1.8) vanish, we obtain

$$(6.1.10) \quad \begin{aligned} \frac{1}{4\pi} \int_{S_*} \frac{P_n g}{R'^{n+1}} dS + \int_K \frac{P_n \kappa}{R'^{n+1}} dV &= \begin{cases} \Phi_s - \omega^2 q_*^2 & \text{for } n=0 \\ \frac{\omega^2}{3} P_2(\bar{\mu}) & n=2 \\ 0 & n=1, 3, 4, 5, \dots \end{cases} \\ + \frac{\omega^2}{2\pi} \int_K \frac{P_n}{R'^{n+1}} dV &= \frac{\omega^2}{3} P_2(\bar{\mu}) \end{aligned}$$

since

$$Q_P = \frac{\omega^2 R^2 \sin^2 \vartheta}{2} = \frac{\omega^2 s_P^2}{2}, \quad P_2(\cos \vartheta) = \frac{3}{2} \cos^2 \vartheta - \frac{1}{2}$$

This set of equations can be completed by an equation which is readily obtained from (6.1.2). If this latter equation is integrated over the whole volume of the liquid mass, the right-hand member

becomes $-4\pi fm + 2\omega^2 V$ where m is the total mass. Since the left-hand member can be transformed into a surface integral by Green's formula, we obtain the equation of Poincaré.¹²

$$\int_S g dS = 4\pi fm - 2\omega^2 V$$

If a similar transformation is applied to (6.1.2) and the volume V_* taken, the equation

$$(6.1.11) \quad \frac{1}{4\pi} \int_{S_*} g dS + f \int_L \kappa dV + \frac{\omega^2}{2\pi} \int_K dV = fm - \frac{2}{3} \omega^2 q_*^3$$

is readily obtained, as a supplementary condition.

6.2. Outlines of Solutions

Written in the form (6.1.10) and (6.1.11) the equations are not yet expressed in terms of those variables which must be determined. In the choice of such variables Wavre's method differs from those discussed in preceding sections. The level surfaces S_* are in this method represented in parametrical form by the equations

$$(6.2.1) \quad x = R \sin \vartheta \cos \psi, \quad y = R \sin \vartheta \sin \psi, \quad z = R \cos \vartheta, \\ R = R(q_*, \vartheta, \psi)$$

Thus, the assumption is made that the set of level surfaces is determined by the parameter q_* . It corresponds to a set of concentric spheres with a radius q_* varying from zero to a certain value given for the free surface of the liquid mass.

Now, the conditions (6.1.10) and (6.1.11) can be transformed to introduce the variables R, ϑ, ψ , but no essential progress can be made with a general form of the function $R(q_*, \vartheta, \psi)$. Therefore, Wavre assumed that

$$(6.2.2) \quad R = q_*(1 + \zeta)$$

¹² From this equation, Poincaré obtained an upper limit for the angular velocity. If the particles on the free surface of a figure of equilibrium are not separated, g has no negative values on S . Calling $\kappa_m = m/V$ the mean density we obtain from the equation $\omega^2 < 2\pi/\kappa_m$. A lower value for the limit ($\omega^2 < \pi/\kappa_m$) under certain conditions concerning the regularity of the surface S has been found by Crudeli [2] and Nikliborc [1]. Crudeli also proved [3], [4] another inequality restricting the highest admissible value of ω . See also Prasad [1].

where the function $\zeta = \zeta(q_*, \vartheta, \psi)$ can be expanded in a series

$$(6.2.3) \quad \zeta = \omega^2 \zeta^{(1)} + \omega^4 \zeta^{(2)} + \dots$$

These assumptions enable us to compare the results that can be expected by applying Wavre's method and those suggested by other investigators. On substituting (6.2.2) in (6.2.1) we obtain a parametrical representation of level surfaces different from (4.1.2) used by Liapounov or (5.1.1) of Lichtenstein. The parameter q_* corresponds to the case of spheroids.

If in (4.1.2) instead of taking unequal axes, we put the radius of the boundary surface equal to $\sqrt{\varrho}$, it follows from (6.2.1) and (6.2.2) that $q_* = a\sqrt{\varrho}$. The equations (4.1.2) represent the level surfaces in the final form adopted by Liapounov after he had tried some others in his first investigations. He realized that this form is very convenient for the exact proof of the existence of figures of equilibrium differing but little from ellipsoids. For the law of density (4.1.1), the function ζ representing the departure of such level surfaces from ellipsoids could be expanded in a series (4.3.19). Thus, Liapounov's results will hold for any angular velocity which will determine the shape of an ellipsoid and for certain small but finite departure from homogeneity since δ is not assumed to have infinitely small values. A similar approach is in Lichtenstein's method in which another parameter (λ) is used. It is defined by the first equation (5.1.11), and this parameter can be used in case of a homogeneous liquid, when δ of Liapounov and of Lichtenstein in the density law is identically equal to zero. Wavre, however, on making use of the series (6.2.3) restricts his considerations to figures which differ but little from a sphere. As mentioned in Section (2.3) only the terms of the order ω^2 were retained in Clairaut's theory and this assumption has been made in several further investigations. On considering this as a first approximation, Wavre made use of the function ζ as represented by the equation (6.2.3). On using the transformed equations (6.1.10) and (6.1.11) he has shown that by successive approximations the set of functions $\zeta^{(1)}$, $\zeta^{(2)}$ (similar to ζ_i of Liapounov's theory) can be expressed in terms of the parameter q_* and of spherical angles, if, of course, the density $\kappa = \kappa(q_*)$ is a given function.

Wavre's function¹³ ζ represents the deformation of a level surface with respect to a sphere of the radius q_* , and the effective calculation of two first approximation yields figures of revolution. By computing the first approximation, Wavre obtained the classical results. We can easily see from (5.1.11) that, if the primary figure of equilibrium is a sphere, i.e., if we put $\omega = 0$, the parameter λ becomes proportional to ω_1^2 , and ω_1 is the actual angular velocity of the figure of equilibrium to be determined. Thus, in this respect, the series (6.2.3) becomes a special case of more general expansions used by Lichtenstein (see, for example (5.2.11)). In Wavre's method, the effective computations are to a certain degree simpler than those in Liapounov's or Lichtenstein's method. However, the latter methods are more general and exact.

6.3. Other Investigations

Certain modifications of the method of Wavre were suggested by Mineo [3] in order to simplify the successive approximations. The potential of a stratified heterogeneous body has been expanded again in a series of even powers of the angular velocity. Other expansions for potential, gravity force, and radius of the free surface of the Earth in terms of Legendre functions were also used by Mineo [1], [4] to determine the figure of a planet when the values of gravity at the free surface are given. The proof of impossibility of a homothetic stratification in a rotating fluid mass in equilibrium given by Wavre was also simplified by Mineo [5]. The case of a strictly ellipsoidal stratification in which the ellipticity varies from the center to the surface was discussed by Wavre in a more recent publication [2] and extended by Dive [3]. An exact ellipsoidal stratification is impossible in a fluid body, if $\kappa = \varphi(\phi)$ and the rotation is permanent. Other results concerning the figures of celestial bodies were obtained by Wavre not in direct connection with his method. We shall mention here only the extension of the theorem of Stokes to the case where the angular velocity is a function of the distance s from the axis of rotation. In this form

¹³ ζ is e in Wavre's notation.

Stokes' theorem is as follows: Newton's potential in the space exterior to a celestial body depends only on the free surface (S) on the angular velocity ω (s) and the total mass (M). Using this theorem, Crudeli [3] derived an integral equation from which the gravity at the surface can be determined.

PART II

Other Invariable or Varying Figures

CHAPTER VII

Zonal Rotation

7.1. General Results Concerning Zonal Rotation

The theory of figures of equilibrium of a fluid mass yielded an explanation of many characteristics of celestial bodies. By the highest degree of precision in mathematical methods used to show the existence of a set of figures of equilibrium, a most solid base has been given to this theory. However, there are phenomena known in celestial bodies which require an extension of the theory for states other than an equilibrium. As mentioned before, the so-called zonal rotation is observed at the present time on the Sun, Jupiter, and Saturn in our solar system. It is well known that in certain nebulae the rotation follows a law different from that holding for the Sun. Faye's law (1880) represents very adequately the

$$(7.1.1) \quad \omega = a + b \sin^2 \vartheta_h \quad (a = 14^\circ.44, \quad b = -2^\circ.31)$$

zonal character of rotation of solar spots up to the heliocentric latitude $\vartheta_h = \pm 35^\circ$ discovered by Carrington. This empirical law applies also to other movements in the atmosphere of the Sun, showing that in general the angular velocity ω is increasing at the surface with the distance of a particle from the axis of rotation. There is no such law given for a similar phenomenon observed on the Jupiter or Saturn.

Many theoretical investigations dealt with the problem of zonal rotation.¹⁴ Initiated to explain Faye's law of the zonal rotation of the Sun, they were extended to various other aspects of the problem as well as to the conditions of a general circulation of matter in a celestial body. The latter problem has been discussed

¹⁴ Most complete lists of these investigations were given by Jardetzky [4] and Wasiutynski [1].

by Jeans [3] and in the work of Wasiutynski [1] in all detail. Such a detailed discussion is beyond the scope of this investigation. However, some further data will be given in Section (7.4). It is important to mention that there is as yet no general agreement about the physical phenomenon which could produce and maintain zonal rotation in a celestial body (see, for example, Jardetzky [13]). Nevertheless, it is quite natural to consider it as a possible second approximation in the case of a rotating fluid mass, whenever the first approximation based on the assumption of a simple equilibrium cannot provide a complete explanation.

In the first attempts to give a theory of zonal rotations, the equations of motion of a viscous fluid were used. Wilsing (in 1891, 1896) suggested the hypothesis that the core of the Sun rotates with a constant angular velocity and that in the outer layer, the distribution of ω is given by the law $\omega = \text{funct.}(s, z)$, where s is the distance from the axis. The possibility of the laws $\omega = \omega(s^2)$ or $\omega = \omega(z, s^2)$ as well as the circular motion in a viscous fluid were investigated also by Sampson (1894), Harzer (1891-97) and Wilczynski (1896-97) (see Jardetzky [4]).

The permanent and zonal rotations in a mass of an ideal fluid were treated by Véronnet [1]-[3], Wavre [1], Dive [1], [2], Jardetzky [1]-[4] and others.

The most important problem discussed in many papers was to find the stratification. In some of these investigations, the conditions were derived under which one could expect the existence of a given stratification. Thus, Harzer reached the conclusion that ellipsoids of revolution can be surfaces of equal density in a viscous fluid having a zonal rotation. (See also, Dive [2], Danoz [1].) The direct way to determine the figure of a fluid mass, when the law of rotation is given, i.e., the angular velocity is expressed in terms of coordinates, has been used by Wavre and Jardetzky.

7.2. Fundamental Equation

The function $Q = (\omega^2 s^2)/2$ introduced in equation (6.1.1) is a particular form of the potential of acceleration as may be easily seen from equations (1.1.2) and (1.1.5). It is, therefore, evident

that the equation

$$(7.2.1) \quad fU + Q = \int \frac{dp}{\kappa} + \text{const.}$$

will hold for the general case where there is a potential of accelerations ($\mathbf{a} = -\text{grad } Q$). As long as there are no other conditions imposed, one can postulate any distribution of accelerations in a fluid mass. For example, we can assume that a rotation occurs according to a law

$$(7.2.2) \quad \omega = \omega(s^2) \text{ or } \omega(s^2, z)$$

In the first case on multiplying equations (1.1.3) by dx, dy, dz respectively and adding them, we obtain the term $-\omega^2 ds^2$ ($s^2 = x^2 + y^2$).

Thus

$$(7.2.3) \quad Q = \int \omega(s^2) ds^2$$

We call the equation of the form (7.2.2) the law of rotation. For any form of this law which differs from that corresponding to the relative equilibrium, we can apply one of the methods discussed in preceding chapters. The problem of figures of masses, the motion of which is subjected to a given law of rotation, can be reduced to a fundamental equation of the type considered before with a single change affecting the term representing the centrifugal force. Thus, for example, on restricting ourselves by the first law (7.2.2), the functional equation, the solutions of which will determine figures of a given fluid mass, is in its general form

$$(7.2.4) \quad fU + \int \omega(s^2) ds^2 = \Phi(\kappa)$$

There are no methods to resolve this equation without some new assumptions. We have to make use of some known representations of the potential U and take a definite law of rotation. If we assume that the distribution of angular velocities in a mass does not differ essentially from a constant value of ω , we can expect that a new figure of the fluid is a slightly deformed figure of equilibrium. Whether such an assumption is correct or not, it should be proved, of course, by a discussion of all conditions of the problem.

One of such conditions is immediately derived from the properties of the fluid. If the density κ is constant or a function of the pressure alone ($\kappa = \varphi(p)$) the equation (1.1.2) shows that the acceleration (7.2.1) has a potential. Now, if the motion assumed is reduced to a pure rotation about a fixed axis (z) it follows from the equations (7.2.2) that

$$\frac{\partial Q}{\partial x} = -\omega^2 x, \quad \frac{\partial Q}{\partial y} = -\omega^2 y, \quad \frac{\partial Q}{\partial z} = 0$$

i.e., ω cannot depend on z . The law of rotation is of the form of the first equation (7.2.2).¹⁵

Wavre and Danoz applied the uniform method discussed in Chapter VI to zonal rotations determined by the law

$$(7.2.5) \quad \omega^2 = a + bs^2$$

where both constants, a and b , have small values. The case considered by Jardetzky is more general because of the form of the law of rotation. He has assumed that

$$(7.2.6) \quad \omega = \omega_0 + \omega_1(s^2, \lambda)$$

where: (a) ω_0 can have any finite value; (b) only the parameter λ takes small values; and (c) the function ω_1 , is such that the function Q can be represented by an absolutely and convergent series

$$(7.2.7) \quad Q(s^2, \lambda) = \frac{\omega_0^2 s^2}{2} + \sum_{k=1}^{\infty} Q_k(s^2) \lambda^k$$

The domain of convergence is given by the conditions

$$(7.2.8) \quad 0 \leq \lambda \leq \lambda_1 \quad 0 \leq s \leq s_m$$

where λ_1 , is a certain limit value and s_m a maximum distance from the axis of a particle at the free surface.

Besides this, the function

¹⁵ For the second law (7.2.2), no equation of the form (7.2.4) will hold (Bjerknes [1]). Attempts were made to determine a law of rotation $\omega(x, y, z)$ such that it will correspond to a given characteristic of density distribution, for example, to an ellipsoidal stratification (method of Dive [2], for permanent rotations).

$$(7.2.9) \quad Q'' = \sum_{k=2}^{\infty} Q_k (s^2) \lambda^k$$

as well as its first derivatives with respect to coordinates should not be larger than $A_1 \lambda_1^2$ where A_1 , is a certain constant.

Obviously, Faye's law (7.1.1) corresponds to the simplest case of (7.2.6), namely, to

$$\omega = \omega_0 + \lambda s^2$$

since ω_0 is not necessarily small. This is also a more general form than the law (7.2.5). Under such assumptions, the figures of a rotating fluid mass which differ but little from an ellipsoid because of the zonal character of rotation could be determined by the method of Liapounov for a homogeneous as well as for slightly heterogeneous masses. Inserting the new value (7.2.7) of the acceleration potential in the equation (4.3.3) of Liapounov's theory, we obtain

$$(7.2.10) \quad \sum_{i=0}^{\infty} U_i + \frac{\omega_0^2}{2f} s^2 + \sum_{k=1}^{\infty} Q_k \lambda^k = \text{funct. } (a)$$

On assuming again that the level surfaces have the form determined by a function ζ and represented by equations (4.1.2) or (4.3.1)–(4.3.2) we can make use of the expansion of Newton's potential in a series (4.1.15). Then, the transformation of this series which yields the equation (4.3.16) applies to (7.2.10). The final result is, then, the equation

$$(7.2.11) \quad R\zeta - \frac{1}{4\pi a^2} \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = \hat{W} + P(a)$$

where

$$(7.2.12) \quad \hat{W} = W + \frac{1}{4\pi a^2 \Delta} \sum_{k=1}^{\infty} Q_k \lambda^k$$

and W is given by (4.3.17). The function $P(a)$ is determined by the formula (4.3.18).

In these equations it cannot be seen whether an ellipsoid of revolution or that with three unequal axis can be used as a figure of reference. The assumption that the rotation is permanent

restricts, however, the choice of the figure of reference (Pizzetti [1]). For a fluid, we have the equation of continuity

$$\frac{d\kappa}{dt} + \kappa \operatorname{div} \mathbf{v} = 0$$

Since the motion is permanent $\partial \mathbf{v} / \partial t = 0$ and $\partial \kappa / \partial t = 0$. Thus, we have

$$\mathbf{v} \cdot \operatorname{grad} \kappa + \kappa \operatorname{div} \mathbf{v} = 0.$$

If the law of rotation is $\omega = \omega(s^2)$ and \mathbf{v} is given by (1.1.1)

$$(7.2.13)^{16} \quad \operatorname{div} \mathbf{v} = -y \frac{\partial \omega}{\partial x} + x \frac{\partial \omega}{\partial y} = \frac{1}{2} \frac{D(s^2, \omega)}{D(x, y)} = 0$$

and we obtain the condition

$$(7.2.14) \quad \mathbf{v} \cdot \operatorname{grad} \kappa = 0$$

The surfaces of equal density must be surfaces of revolution.

The function ζ which yields the stratification in the fluid mass must satisfy the equation (7.2.11) and the condition (4.1.6)

$$(7.2.15) \quad \int \zeta d\sigma = - \int \zeta^2 d\sigma - \frac{1}{3} \int \zeta^3 d\sigma$$

This condition is necessary in order to determine the arbitrary function of the parameter a introduced in the expression of P .

7.3. Figures That Differ But Little From Ellipsoids

To solve the equation (7.2.11) we assume that ζ can be expanded in the absolutely and uniformly convergent series

$$(7.3.1) \quad \zeta = \sum_{i+j>0}^{\infty} \zeta_{ij} \delta^i \lambda^j$$

where ζ_{ij} are functions of a and ϑ . Since the right-hand member of (7.2.11) depends also on both parameters δ and λ , it has to be expanded in a series similar to (7.3.1), and we put, therefore

$$(7.3.2) \quad \hat{W} = \sum \hat{W}_{ij} \delta^i \lambda^j$$

¹⁶ Thus, the equation (7.2.13) is necessarily satisfied by an incompressible fluid, but the inverse statement is not true.

where \hat{W}_{ij} are functions of a and ϑ ($\hat{W}_{00} = 0$). On inserting (7.3.1) and (7.3.2) in (7.2.11), we obtain a set of equations

$$(7.3.3) \quad R\zeta_{ij} - \frac{1}{4\pi a^2} \int \frac{\bar{\zeta}'_{ij} d\sigma'}{D(a, 1)} = \hat{W}_{ij} + f_{ij}(a)$$

which must be satisfied by the coefficients ζ_{ij} . The functions $f_{ij}(a)$ are unknown but another set of equations derived from (7.2.15) will help us to find these functions. We have, namely,

$$\int \sum \zeta_{ij} \delta^i \lambda^j d\sigma = - \int (\sum \zeta_{ij} \delta^i \lambda^j)^2 d\sigma - \frac{1}{3} \int (\sum \zeta_{ij} \delta^i \lambda^j)^3 d\sigma$$

Hence

$$(7.3.4) \quad \int \zeta_{10} d\sigma = 0, \quad \int \zeta_{01} d\sigma = 0, \quad \int \zeta_{20} d\sigma = - \int \zeta_{10}^2 d\sigma, \dots$$

Each integral

$$(7.3.5) \quad N_{ij} = \int \zeta_{ij} d\sigma$$

can be expressed in terms of those functions ζ_{kl} for which $k + l$ is less than $i + j$. Thus, N_{ij} can be subsequently computed.

If (7.3.3) is multiplied by $d\sigma$ and integrated over the unit sphere, the equation

$$(7.3.6) \quad RN_{ij} - \frac{1}{4\pi a^2} \int \bar{\zeta}'_{ij} d\sigma' \int \frac{d\sigma}{D(a, 1)} = \int \hat{W}_{ij} d\sigma + 4\pi f_{ij}(a)$$

will be of the form (4.3.23). On taking into account the value (4.3.27) of the second integral in the second term and eliminating f_{ij} from (7.3.3) and (7.3.6) we obtain the equation

$$(7.3.7) \quad R\zeta_{ij} - \frac{1}{4\pi a^2} \int \bar{\zeta}'_{ij} \left(\frac{1}{D(a, 1)} - \frac{1}{a} \tan^{-1} \frac{1}{\sqrt{\varepsilon}} \right) d\sigma' \\ = \hat{W}_{ij} - \frac{1}{4\pi} \int \hat{W}_{ij} d\sigma + \frac{1}{4\pi} RN_{ij}$$

Now, the expression \hat{W} (7.2.12) begins with the term of the order ζ^2 . The coefficients \hat{W}_{ij} will therefore depend only on those functions which have subscripts subjected to the same condition $k + l < i + j$ we had for N_{ij} .

In the equation (7.3.7) we have the function ζ_{ij} which is to be found for the whole range of values of the parameter a and the angle ϑ . In the second term the value $\bar{\zeta}'_{ij}$ is involved. It is taken for $a = 1$, i.e., for the free surface of the fluid. Thus, the natural way to determine this function is to put in (7.3.7) $a = 1$ again, to solve the integral equation

$$(7.3.8) \quad R\bar{\zeta}_{ij} - \frac{1}{4\pi} \int \frac{\bar{\zeta}'_{ij} d\sigma'}{D(1, 1)} = \bar{W}_{ij} + \text{const.}$$

and to substitute the function $\bar{\zeta}_{ij}$ into (7.3.7). The bar denotes as before the value of an expression for $a = 1$. It is easy to see which terms in (7.3.7) after the substitution $a = 1$ yield the constant term in (7.3.8). N_{ij} are represented by the integrals (7.3.5) and are functions of a only. The same holds for terms $-1/4\pi \int \bar{W}_{ij} d\sigma$. Now, to transform the third term of the left-hand member of (7.3.7) for $a = 1$, one has to put in (4.3.28) $a = 1$ and $\varrho = t = \varepsilon$; this term is then

$$\frac{1}{4\pi} \int \bar{\zeta}'_{ij} \tan^{-1} \frac{1}{\sqrt{\varrho}} d\sigma' = \text{const.} \int \bar{\zeta}'_{ij} d\sigma' = \text{const.} \bar{N}_{ij} = \text{const.}$$

The function ζ (7.2.16) can be, therefore, calculated by successive approximations on resolving a double set of equations (7.3.7) and (7.3.8) which yield the functions

$$(7.3.9) \quad \zeta_{ij} = \zeta_{ij}(a, \vartheta), \quad \bar{\zeta}_{ij} = \zeta_{ij}(1, \vartheta)$$

According to the first conditions (7.3.4) we have

$$(7.3.10) \quad N_{10} = N_{01} = 0$$

By (7.2.12) and (4.3.17) we can easily see, that

$$(7.3.11) \quad \begin{aligned} \bar{W}_{10} &= \frac{1}{4\pi a^2 \Delta} [\Phi_0 - \Phi_0(0)] \\ \bar{W}_{01} &= \frac{1}{4\pi a^2 \Delta} Q_{01}(a, \vartheta) \end{aligned}$$

The latter expression is obtained if we make use of the fact that Q_k in (7.2.12) are functions of s^2 . Because of $s^2 = x^2 + y^2 =$

$a^2(1 + \zeta)^2(\varrho + 1) \sin^2 \vartheta$ on a level surface Q_k depend on ζ , i.e., on the parameter δ too. Thus, we have to put

$$(7.3.12) \quad \sum_{k=1}^{\infty} Q_k \lambda^k = \sum_{\mu=0}^{\infty} \sum_{\nu=1}^{\infty} Q_{\mu\nu} \delta^{\mu} \lambda^{\nu}$$

As in the problem of the figures of equilibrium, the expressions Φ_0 and $\Phi_0(0)$ are obtained by substituting $\varkappa' = \varphi(a')$ in the corresponding terms of Newton's potential U (see, (7.2.15)) and (4.1.17). The effective calculation gives then (Jardetzky [4])

$$(7.3.13) \quad W_{10} = \frac{1}{a^2} \int_0^a \varphi(a') a' (\tan^{-1} \sqrt{\varrho} - \tan^{-1} \sqrt{\varepsilon}) da'$$

where ε' is the positive root of the equation in t

$$(7.3.14) \quad \frac{\varrho + 1}{t + 1} \sin^2 \vartheta + \frac{\varrho}{t} \cos^2 \vartheta = \left(\frac{a'}{a} \right)^2$$

In general, if the density law, i.e., the function $\varphi(a)$ is given, it is possible to expand W_{10} in a series of spherical functions with coefficients which will depend on a . It was shown by Liapounov [9] (Part I, pp. 27–34 and 66–98) that under these conditions, the function ζ_{10} can be represented by the series of the form

$$(7.3.15) \quad \zeta_{10} = \sum Z_{2n}$$

where the spherical function Z_{2n} is given by the equation

$$(7.3.16) \quad Z_{2n} = \frac{P_{2n}(\bar{\mu})}{\gamma_{2n}} \int W_{10} P_{2n}(\bar{\mu}) d\sigma$$

$$\gamma_{2n} = \frac{4\pi}{4n + 1}, \quad \bar{\mu} = \cos \vartheta$$

and $P_{2n}(\bar{\mu})$ are Legendre polynomials. Jardetzky [3] has proved that the function ζ_{01} can be represented by the series

$$(7.3.17) \quad \zeta_{01} = \sum \beta_{2n} P_{2n}(\bar{\mu})$$

and the last two results show the difficulty and extremely long calculations needed to find the expression ζ_{ii} in terms of coordinates.

If we restrict ourselves by the first approximation

$$(7.3.18) \quad \zeta_1 = \zeta_{10} \delta + \zeta_{01} \lambda$$

we can make use of two known general expressions (7.3.15) and (7.3.17) on substituting certain particular laws of density and rotation. Liapounov calculated ζ_{10} in the case where $\varphi(a)$ is a polynomial in a^2 of degree k . In order to find a figure of a fluid mass in case of a zonal rotation Jardetzky made use of the density law

$$(7.3.19) \quad \kappa = 1 + \delta\varphi(a) = 1 + \delta(H_1 a^2 + H_2)$$

where H_1 and H_2 are constants, and assumed that

$$(7.2.6') \quad \omega = \omega_0 + \lambda s^2$$

i.e., that the law of rotation is of the type of Faye's law. On taking the single term (7.3.18), one assumes obviously that the parameters δ and λ are of the same order. Otherwise the terms in δ and λ should be combined in a different manner. After extensive transformations (Jardetzky [4] pp. 48-59) the equation of the free surface of the liquid mass which in general is given by the formula (4.3.2) could be written in the form

$$(7.3.20) \quad \frac{x^2 + y^2}{\varrho + 1} + \left(\frac{1}{\varrho} - 2L_2\delta - K_2\lambda \right) z^2 - (2L_4\delta + K_4\lambda) z^4 = 1 + 2L_0\delta + K_0\lambda$$

where L_i, K_i are constants; $\sqrt{\varrho + 1}$, $\sqrt{\varrho}$, and $\sqrt{\varrho}$ are the semi-axes of the ellipsoid from which the figure (7.3.20) differs but little because of the heterogeneity of the liquid and of the zonal rotation represented by the laws (7.3.19) and (7.2.6'). The constants L_i, K_i can be expressed in terms of ϱ, H_1, H_2 , and ω_0 , but we omit here these complicated formulae.

From (7.3.20) we see that the free surface is represented in a first approximation by an algebraic surface of a simple shape.

The density is a linear function of a^2 decreasing from its value $\kappa_0 = 1 + \delta H_2$ at the center to $\kappa_1 = 1 + \delta(H_1 + H_2)$ at the free surface ($H_1 < 0$, if $\delta > 0$). Maybe this law is too simple to be applied to celestial bodies, but it does not mean that a^2 is a linear

function of the distance from center. It is possible that it would be better to find the connection between a and q^* .

The characteristics of the surface (7.3.20) depend on the numerical values of constants involved. The free surface will be either compressed or stretched along the equator of the ellipsoid of reference according to the conditions ($z = 0$)

$$(7.3.21) \quad 2L_0 \delta + K_0 \lambda \lesseqgtr 0$$

To see the deformation at the poles, one has to compute $\bar{\zeta}_1^*$, at $\vartheta = 0$ and $\vartheta = \pi$. For a negative $\bar{\zeta}_1^*$, the figure (7.3.20) will be compressed at the poles since according to (4.1.2) we have

$$(7.3.22) \quad [z]_{\vartheta=0,\pi} = \pm \sqrt{\varrho}(1 + \bar{\zeta}_1^*)$$

The shape of the meridian is easily seen from (7.3.20).

7.4. Connection Between Zonal Rotation and Convective Currents

The hypothesis of a zonal rotation of the kind considered in preceding sections represents also a certain simplification of the state of motion which is actually observed in many celestial bodies. It is highly probable that such rotation is only one part of a general circulation of matter in a celestial body due to rotation as well as to convective currents. In this section the broad field of internal movements in stars will be discussed only in brief, since as mentioned before, our aim is just to elucidate the part played by the hypothesis of zonal rotation in a very advanced period of the life of a celestial body, namely, during its solidification. In order to be able to develop a mathematical theory for this period, we can, of course, consider the zonal rotation as a given form of motion determined by certain initial conditions. Actually, a similar assumption has been made when the problem of a uniform rotation of a fluid mass has been formulated and solved. Nevertheless, for a better understanding of the evolution of a celestial body, the causes of a zonal rotation as well as the physical phenomena which are able to maintain this kind of motion should be determined.

We have mentioned some earlier investigations which deal with

zonal rotation as a phenomenon characterized by Faye's law of the rotation of the sun. The general problem of the internal constitution and internal movements in stars has not yet been solved, but there are several models which approximate the actual state with more or less precision. The simplest among these models is, of course, that of a gaseous star in an absolute equilibrium. The stratification is then represented by the density expressed in terms of the radius. The factors involved the pressure and the density, are connected by a certain differential equation. Then, some solutions of it have been found for different cases of an adiabatic equilibrium (Emden [1]). The assumption usually made is that in this case, i.e., in the case of a "convective equilibrium" the heat transfer by convective currents from the interior to the surface of the star is included in the general state of the body. But other factors must be taken into account, and Jeans [2] pointed out the role played by the "viscosity of radiation." According to one of his conclusions (Jeans [3] p. 91), there will be no convection currents in a model corresponding to a configuration of equilibrium of the type discussed except maybe near the surface.

We do not discuss here the sources of energy in celestial bodies. The discovery of nuclear processes overshadowed all previous considerations about such sources. Models concerning rotating stars are obviously more important than a static model. Jeans, in the paper just mentioned, [2], has shown that the inner parts of a star must rotate more rapidly than the outer layer. One of his conclusions (Jeans, [3] p. 265) deserves our special attention. In his opinion, namely, throughout the whole life of any star whatever, "the equalization of angular velocity produced by viscosity is negligible in the central regions of the star." In a simplified model he shows that, if the angular velocity of rotation at an initial moment depends on the radius only, i.e., if the rotation occurs by spherical shells, it will not be destroyed by the regular viscosity or by the radiation. In the second approximation, however, Jeans considered Faye's law of rotation. The fact that the ellipticity of spheroidal layers increases towards the center should yield, namely, the explanation of the equatorial acceleration. The factor pro-

ducing this phenomenon is the drag exerted by inner layers. Thus, according to Jeans, a zonal rotation is not necessarily connected with convective currents. However, the model of spherical shells each rotating with its own angular velocity seems to be too simple an approximation. Among the general assumptions concerning the Sun, we will find the conclusions based on numerous observations as follows (V. Bjerknes [1]): (a) the Sun is near a state of stable internal equilibrium, (b) conditions in general seem to be favorable for the formation of stratified circulation, (c) the origin of all that happens on the Sun is in the solar radiation, (d) heat lost by the photosphere is restored from the deeper layers by internal radiation, convection, and conduction, (e) internal radiation is believed to be the dominant factor especially in the deep layers of very high temperature, (f) the importance of convection increases near the surface, and (g) the rotation of the Sun tends to produce general zonal symmetry.

Obviously, all these facts are beyond the limits of the theories exposed in preceding sections. We could introduce in the equations of motion the law of rotation of the form $\omega = \omega(s^2)$ and apply one of the methods to solve the resulting integral equations. But taking into account new physical factors just mentioned, the equations of motion have to be modified as well as supplemented by a set of new conditions. Under new assumptions it is, for example, shown (Krogdahl [1]) that if a celestial body is composed of a viscous fluid its rotation would give rise to meridional currents, which would cause the zonal character of rotation. In Bjerknes' theory the existence of such currents was just postulated.

We know several special conditions of equilibrium derived for different kinds of physical phenomena. For a stable equilibrium of an incompressible ideal fluid when none but mechanical phenomena are considered the condition

$$\frac{d(\omega r^2)}{dr} \geq 0$$

must hold as proved by Rayleigh in 1917.¹⁷ It concerns the cir-

¹⁷ See also Randers [1].

culation along any parallel. An extension of this condition is found in the Solberg-Hoiland criterium of stability for compressible fluids (Wasiutynski [1]). The formation of large-scale currents could result in case of gravitational instability. However, if the conditions of gravitational stability are satisfied, the convective current may arise because of the fact that the gravitationally stable layers are not necessarily also in a thermal equilibrium (Vogt, Eddington [1], Jeffreys [1]). In such a case the resulting pressure gradient on a level surface would tend to make the matter flow from equator to poles, or *vice versa*. The equilibrium could be maintained if the temperature distribution does not give rise to this component of the pressure gradient, but when the radiation is also taken into account, this will lead to a new form of conditions of equilibrium. According to v. Zeipel's theorem [1] the generation of energy $4\pi E$ (per second and gram) within a rotating gaseous mass must follow the law

$$4\pi E = B \left(1 - \frac{\omega^2}{2\pi f \kappa} \right)$$

where B is a constant. It was pointed out by Eddington [1] that the generation of energy from nuclear sources "can scarcely follow a law of this kind." Among possible internal movements which would result, since the equilibrium cannot be realized, the primary currents in planes through the meridians seem to be the most likely phenomenon according to Eddington. These currents must be, then, deflected (East or West) by the rotation of the body, and different periods of rotation will result in different latitudes and at different depths. In this general characteristic both cases are included—an angular velocity increasing in a layer with the increasing distance from the axis of rotation as well as a decreasing angular velocity.

The regular viscosity and the radiation (or the radiative viscosity) are obviously factors acting in opposite directions. The former tends to even out the differences in angular velocities while the latter can establish currents and, therefore, such differences. From the equations of motion of a viscous fluid Eddington [2]

derived that the currents which will be maintained by the radiation in the interior of the mass will correspond to the circulation near the surface from the poles to the equator.

The actual complexity of the problem of convection in celestial bodies is, of course, greater than it was assumed in the cases in which an approximate solution was attempted. Even some conclusions drawn from earlier observations seem to be taken in too simple a form. For example, Tuominen [1] found that the sunspots at latitudes lower than 16° have a drift towards the equator while the spots at higher latitudes move towards the poles. Thus, there can be two opposite directions in the surface layer for convection currents and there are no reasons to exclude the existence of a still more complicated distribution of velocities.

At the present stage of the theory of internal motion in celestial bodies we must point out that as long as pure dynamics is involved, both cases, the angular velocity increasing or decreasing with the increasing distance from the axis of rotation, are equally possible. In many investigations in which the problem of convective currents is considered, under special conditions a decreasing angular velocity resulted. Nevertheless, sometimes an angular velocity increasing in a layer near the free surface from poles to equator was also derived. As to the actual state in celestial bodies, the outer layers of planetary nebulae are rotating usually less rapidly than the core. Zonal rotation is characterized in the case of the Sun, Jupiter, and Saturn by an opposite law. The well-known experiments of Belopolsky and Riaboushinsky showed that, if in a glass sphere filled with a liquid differences in angular velocity are produced, the currents along the axis of rotation towards the center and moving along an equatorial radius from it are connected with an angular velocity decreasing as the distance from the axis of rotation is increasing. These currents and those considered by Eddington have opposite directions. In the experiments just mentioned, there is a new factor involved, namely, the friction at the surface of the fluid which obviously does not play any part in stars. It seems, however, that the effect of it does not change essentially the general character of internal movements, and the

experiments of Belopolsky [1] and Riaboushinsky [1] illustrate well the zonal rotation.

On the other hand these experiments correspond better to conditions for convective currents which one can expect to find in the fluid core of a planet or of a star already covered by a solid crust. Then, the friction between a very viscous liquid interior and the crust can become an important factor.

The studies of such an advanced stage as a solidifying celestial body will certainly meet new theoretical difficulties, since, for example, the form of the equation of state for plastic and solid bodies is much more difficult to determine than that for gases or liquids.

A very important general conclusion has been reached by Joukowski [1] in the problem of motion of a rigid body having a cavity filled with a viscous liquid. In this problem the friction at the walls of the cavity was also taken into account. According to Joukowski if there are convection currents in the viscous liquid in the cavity, they will be directed from poles towards the equator along the surface when the mean square of angular velocity is a function increasing with the distance from the axis of rotation. The opposite distribution of angular velocities will take place if the direction of convective currents is reversed.¹⁸

7.5. Zonal Rotation in a Core

In order to study the conditions of the problem of internal movements in a viscous core of a planet, we shall write in a vector notation the equations used by Joukowski [1]. To simplify the problem an incompressible liquid is considered, and, therefore, we have

$$(7.5.1) \quad \operatorname{div} \mathbf{v} = 0$$

where \mathbf{v} is the velocity. Let $\boldsymbol{\Omega}$ be the angular velocity of a particle. Since $\operatorname{curl} \mathbf{v} = 2\boldsymbol{\Omega}$ and $\operatorname{curl} \operatorname{curl} \mathbf{v} = \operatorname{grad} \operatorname{div} \mathbf{v} - \nabla^2 \mathbf{v}$, we obtain

¹⁸ This statement is sometimes quoted in the literature in a not quite exact form. The paper of Joukowski [1] has been published in Russian.

$$(7.5.2) \quad \nabla^2 \mathbf{v} = -2 \operatorname{curl} \boldsymbol{\Omega}$$

Now, the equation of motion of a viscous liquid is

$$(7.5.3) \quad \mathbf{a} = \mathbf{F} - \frac{1}{\varkappa} \operatorname{grad} p + \nu \nabla^2 \mathbf{v}$$

We assume: (a) that $\mathbf{F} = \operatorname{grad} U$, i.e., is derived from Newton's potential, (b) the kinematical coefficient of viscosity is constant, and (c) the liquid is homogeneous ($\varkappa = \text{const.}$). The differential equations of motion can be written in terms of components of the angular velocity, and the Helmholtz equations represent such a form for perfect fluid. It is known that to obtain these equations one has to apply the operation curl to (7.5.3). Then, if

$$\operatorname{curl} \mathbf{F} = 0, \quad \operatorname{curl} \frac{1}{\varkappa} \operatorname{grad} p = 0$$

we obtain

$$(7.5.4) \quad \frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega} \cdot \nabla) \mathbf{v} + \frac{\nu}{2} \operatorname{curl} \nabla^2 \mathbf{v}$$

To transform this equation we make use of the known formulae as follows

$$(7.5.5) \quad \begin{aligned} \frac{d\boldsymbol{\Omega}}{dt} &= \frac{\partial \boldsymbol{\Omega}}{\partial t} + (\mathbf{v} \cdot \nabla) \boldsymbol{\Omega} \\ \operatorname{curl} [\mathbf{v} \times \boldsymbol{\Omega}] &= \nabla \times [\mathbf{v} \times \boldsymbol{\Omega}] \\ &= \mathbf{v}(\nabla \cdot \boldsymbol{\Omega}) + (\boldsymbol{\Omega} \cdot \nabla) \mathbf{v} - \boldsymbol{\Omega}(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla) \boldsymbol{\Omega} = (\boldsymbol{\Omega} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \boldsymbol{\Omega} \end{aligned}$$

because of

$$\nabla \cdot \mathbf{v} = \operatorname{div} \mathbf{v} = 0, \quad \nabla \cdot \boldsymbol{\Omega} = \frac{1}{2} \operatorname{div} \operatorname{curl} \mathbf{v} = 0$$

Now, by (7.5.4), (7.5.2), (7.5.5) and putting

$$(7.5.6) \quad \operatorname{curl} \boldsymbol{\Omega} = 2\boldsymbol{\Omega}_1 \quad \operatorname{curl} \boldsymbol{\Omega}_1 = 2\boldsymbol{\Omega}_2$$

we obtain

$$(7.5.7) \quad \frac{\partial \boldsymbol{\Omega}}{\partial t} = \operatorname{curl} [\mathbf{v} \times \boldsymbol{\Omega}] - 4\nu \boldsymbol{\Omega}_2$$

Thus, the forces are eliminated from the equations, and the initial and boundary conditions for the liquid part must now be

given. Joukowski made use of a simple form of boundary conditions. They express the fact that friction acting at a point of the internal boundary surface of the crust in the direction of the relative velocity (v_r) of the liquid is balanced by the internal friction of the core. If τ is the coefficient of external friction at the boundary of the liquid and μ the coefficient of viscosity, we have the first boundary condition in the form

$$(7.5.8) \quad \tau v_r = \mu \frac{\partial v_r}{\partial n}$$

The second one must express the continuity of the normal velocity component at the boundary of the cavity

$$(7.5.9) \quad [v_n]_{\text{core}} = [v_n]_{\text{crust}}$$

The general scalar equations which can be obtained from the vector equation (7.5.7)¹⁹ were used by Bondi and Lyttleton [1] in an investigation concerning the internal movements in a liquid core. One of their conclusions is of great importance at least in case of the Earth, and we shall present an abbreviated form of the mathematical analysis of the problem.

Different generalized orthogonal coordinates q_1, q_2, q_3 can be used and the corresponding scalar equations derived from (7.5.7). On representing the line element by the formula

$$ds^2 = S_1 dq_1^2 + S_2 dq_2^2 + S_3 dq_3^2$$

and denoting by v_1, v_2, v_3 and $\omega_1, \omega_2, \omega_3$ the components of velocity \mathbf{v} and angular velocity $\mathbf{\Omega}$ respectively in directions corresponding to q_1, q_2, q_3 , we write the first equation of motion, which follows from (7.5.7) in the form

$$(7.5.10) \quad \begin{aligned} \frac{\partial \omega_1}{\partial t} = & \frac{1}{S_2 S_3} \left[\frac{\partial}{\partial q_2} S_3 (v_1 \omega_2 - v_2 \omega_1) - \frac{\partial}{\partial q_3} S_2 (v_3 \omega_1 - v_1 \omega_3) \right] \\ & - \frac{\nu}{S_2 S_3} \left\{ \frac{\partial}{\partial q_2} \frac{S_3}{S_1 S_2} \left[\frac{\partial}{\partial q_1} (S_2 \omega_2) - \frac{\partial}{\partial q_2} (S_1 \omega_1) \right] \right. \\ & \quad \left. - \frac{\partial}{\partial q_3} \frac{S_2}{S_3 S_1} \left[\frac{\partial}{\partial q_3} (S_1 \omega_1) - \frac{\partial}{\partial q_1} (S_3 \omega_3) \right] \right\} \end{aligned}$$

¹⁹ Jardetzky [10].

The second and third equation are readily obtained by changing the subscripts. For further investigations the cylindrical coordinates $q_1 = z$, $q_2 = r$, $q_3 = \gamma$ are used, and we have $ds^2 = dz^2 + dr^2 + r^2 d\gamma^2$. Hence $S_1 = S_2 = 1$, $S_3 = r$. Bondi and Lyttleton introduced an operator to simplify the form of equations and for the cylindrical coordinates this operator is defined as²⁰

$$(7.5.11) \quad \hat{D}^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \equiv \frac{\partial^2}{\partial z^2} + r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r}$$

It can be used in case of axial symmetry. Assuming that all the factors involved in (7.5.10) do not depend on the angle γ (longitude) these equations can be reduced to a much simpler form. From the expression of $\text{div } \mathbf{v}$ in terms of generalized coordinates

$$\text{div } \mathbf{v} = \frac{1}{S_1 S_2 S_3} \left[\frac{\partial}{\partial q_1} (v_1 S_2 S_3) + \frac{\partial}{\partial q_2} (v_2 S_3 S_1) + \frac{\partial}{\partial q_3} (v_3 S_1 S_2) \right]$$

it may be easily seen that the equation (7.5.1) for cylindrical coordinates and axial symmetry will be satisfied if we put

$$(7.5.12) \quad rv_1 = \frac{\partial \psi}{\partial r} \quad rv_2 = -\frac{\partial \psi}{\partial z}$$

The well-known formulae for the components of $\text{curl } \mathbf{v} = 2\boldsymbol{\Omega}$ yield the expressions

$$(7.5.13) \quad \begin{aligned} \omega_1 &= \frac{1}{2S_2 S_3} \left[\frac{\partial}{\partial q_2} (S_3 v_3) - \frac{\partial}{\partial q_3} (S_2 v_2) \right] \\ \omega_2 &= \frac{1}{2S_3 S_1} \left[\frac{\partial}{\partial q_3} (S_1 v_1) - \frac{\partial}{\partial q_1} (S_3 v_3) \right] \\ \omega_3 &= \frac{1}{2S_1 S_2} \left[\frac{\partial}{\partial q_1} (S_2 v_2) - \frac{\partial}{\partial q_2} (S_1 v_1) \right] \end{aligned}$$

On putting

$$(7.5.14) \quad S_3 v_3 = rv_3 = \chi$$

the expressions (7.5.13) take the form (for cylindrical coordinates)

²⁰ The symbol $\hat{}$ is used to avoid the confusion with the distance D in preceding chapters.

$$(7.5.15) \quad \omega_1 = \frac{1}{2r} \frac{\partial \chi}{\partial r} \quad \omega_2 = -\frac{1}{2r} \frac{\partial \chi}{\partial z} \quad \omega_3 = \frac{1}{2} \left[\frac{\partial v_2}{\partial z} - \frac{\partial v_1}{\partial r} \right]$$

By (7.5.12), (7.5.15), and (7.5.11) we have

$$(7.5.16) \quad \omega_3 = -\frac{1}{2r} \left[\frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \psi}{\partial r} \right] = -\frac{1}{2r} \hat{D}^2 \psi$$

Thus, six unknown variables v_i , ω_i are expressed in terms of two functions $\psi(z, r, t)$ and $\chi(z, r, t)$. To determine these functions, we can make use of the first and third equations (7.5.7) on substituting (7.5.12), (7.5.14), (7.5.15), and (7.5.16). These equations were linearized by Bondi and Lyttleton. They obtained first the equations

$$(7.5.17) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \nu \hat{D}^2 \right) \chi - \frac{1}{r} \left[\frac{\partial \psi}{\partial z} \frac{\partial \chi}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial \chi}{\partial z} \right] &= 0 \\ \left(\frac{\partial}{\partial t} - \nu \hat{D}^2 \right) \hat{D}^2 \psi + \frac{2\chi}{r^2} \frac{\partial \chi}{\partial z} &= 0 \end{aligned}$$

and assumed that the motion in the liquid core differs but slightly from the rotation of a rigid body around an axis fixed in space. Under this condition let

$$(7.5.18) \quad \chi = (n + kt)r^2 + \varphi(r^2, z)$$

where n and k are two constants and φ a function having small values of the order of ψ . Bondi and Lyttleton considered the action of tidal friction in the case of the Earth and assumed also that n has a small value while $k < 0$ and $|k| \ll n^2$. We make similar assumptions which will correspond to very small changes in ω_1 , ω_2 , ω_3 . Then, on neglecting in (7.5.7) the products of other small factors and even the term having a factor kt , the equations (7.5.7) become

$$(7.5.19) \quad \begin{aligned} \nu \hat{D}^2 \chi + 2n \frac{\partial \psi}{\partial z} &= kr^2 \\ \left(\frac{\partial}{\partial t} - \nu \hat{D}^2 \right) \hat{D}^2 \psi + 2n \frac{\partial \varphi}{\partial z} &= 0 \end{aligned}$$

To simplify more these linearized equations a very slow depar-

ture from the rotation of a rigid body is considered, the function φ is substituted in the first term of (7.5.19), and the time variation of ψ is neglected. Then, we have

$$(7.5.20) \quad \begin{aligned} \nu \hat{D}^2 \varphi + 2n \frac{\partial \varphi}{\partial z} &= k r^2 \\ \nu \hat{D}^4 \psi - 2n \frac{\partial \psi}{\partial z} &= 0 \end{aligned}$$

Obviously the assumptions which made possible the derivation of (7.5.20) can hardly be adopted for gaseous bodies like stars. They seem to be plausible nevertheless when we consider very slow movements, for example, in the interior of the Earth during different geological periods.

The solutions of (7.5.20) obtained by Bondi and Lyttleton represent those internal movements for which the speed of circulation about the axis of rotation is of a higher order of magnitude than the speed of circulation in meridian planes, i.e., the former is much larger than the latter. This is a conclusion, very important, for the theory of the Earth's figure, since according to the hypothesis of zonal rotation, the principal features of the Earth's crust were produced by internal movements of similar kind.

7.6. Internal Movements and Solidification of a Celestial Body

There are but few planets the surface features of which can be observed, and the structure of the surface layer of only the Earth and the Moon is known in more detail. The variety of causes which can affect the formation of a crust on a cooling celestial body can result, of course, in many different structures. The varying field of forces of attraction and resulting tidal action, the falling meteors or the fission of a body, the nonuniform distribution of matter may play a more or less important part in deformation of a solidifying outer layer of a body. Special attention, however, must be paid to internal movements as to one of the most important factors acting during the formation of a solid crust on a celestial body.

To take into account such internal movements one has to assume that the interior of a planet displayed the properties of a fluid. Whether the usual assumption that the planets were formed from a very hot fluid which was cooling during the whole evolution or the suggestion that the Earth, for example, has formed a liquid core during geologic time is correct is not essential for the hypothesis of internal movements. The latter possibility, namely, that the Earth's core is formed from a state of a nearly uniform mixture of the iron and silicate phases and that in the earlier period of its existence our planet increased its temperature by contraction of the gas and dust mass is suggested by Urey [1]. We do not specify here the chemical and physical processes in the Earth's interior, but at the present time the theory as well as observations reduced the probability of existence of a celestial body without currents of matter in its interior practically to zero. The condition of a rigid-body rotation of any celestial body has been very useful in the theory as long as an explanation of figures on a large scale was gained. The statement that a factor like viscosity could even out all the differences in velocities before a planet solidified was always an assumption suggested only by the lack of knowledge about the sources which could maintain these differences. As mentioned above, the discovery of sources of energy such as the carbon cycle of nuclear reaction made this assumption still less probable than it was earlier. The single criterion in this case must be taken from the observation. If there are no traces of deformations in the solidified portion of a celestial body the internal motion in liquid parts obviously did not play an important part. If the structure of the crust shows that there were certain displacements we cannot yet assert with certainty that they are due exclusively to the action of internal movements. Several theories involving the existence of currents in the Earth's interior were suggested. O. Fisher seems to have been the first to point out in 1889 that convection currents could produce certain tectonic phenomena. This idea was applied in different forms by Ampferer, Andrée, Schwinner, Holmes, Griggs, a.o. Solutions of particular problems were obtained dealing with thermal disturbances like the difference in tempera-

ture distribution under the continents and suboceanic parts of the crust or with cellular convection of Bénard type. According to Urey [1], the assumption that the cooling at the poles or beneath the oceans is the single cause that produces the currents in the mantle leading to mountain chains folding does not appear possible. He admits, however, (Urey [2]) the possibility of a planetary convection which could occur at the time of a high viscosity in the Earth. Jardetzky [13] pointed out that none of these theories of subcrustal currents were able to explain the characteristic facts known about the formation of the Earth's crust. In general, however, according to Vening-Meinesz's opinion [1] the best hypothesis explaining the geological processes has to make use of currents in plastic subcrustal layers. These currents could be produced by the temperature-gradient maintained by other sources of energy or due to internal rearrangement of bodies of different densities. Urey [1] also pointed out the role of such rearrangements. On discussing the problem of convection in a sphere, as well as in the mantle and representing the distribution of current system by spherical harmonics of first five orders, Vening-Meinesz [2] makes these currents responsible for the formation of the protocontinent and for its later distortion, i.e., reaches the same conclusion which has been made by the author for the first time in 1929. (See also, Gutenberg [1]). The problem of thermal instability in a planet heated within is considered again by Chandrasekhar [4]. In order to understand the generation of convection currents, we should know in a more exact way in which factors produce such differences in temperature in any celestial body as well as in the Earth. The simplest assumptions we used before are as follows: in an advanced stage of a celestial body at which the formation of the solid crust began, one can expect that the differentiation of substances is more or less completed, the stratification corresponding to the distribution of forces is settled, and the temperature field is determined by the condition of a thermal equilibrium. By rejecting the latter assumption, the way was open to locate convection currents at any part of a celestial body where they were desired. This explains different aspects of the hypoth-

esis of convection currents and also possibly that fact that none of them are generally accepted. However, a great deal of truth seems to be in this hypothesis. It should be given, therefore, such form that the conclusions drawn from it can be verified in a more precise way than it has usually been done.

The fact that internal movements could occur in every celestial body until it is solidified is self-evident. These movements could possibly have an arbitrary distribution at some earlier period. Taking into account the effect of other factors, one can expect, however, that a certain regular distribution has been imposed on these currents later. It will be very difficult to give a mathematical theory which will prove that a certain type of internal movements *must* occur. In many cases we should be satisfied by a proof that this type of movement is possible. The investigation of Bondi and Lyttleton provided such a proof for the zonal rotation in a planet. As it was shown in the preceding section, in a system of convection currents having an axial symmetry, the circulation about the axis can be the dominant part while the displacements in meridians can be neglected. It must be pointed out that this character of internal movements is not necessarily imposed on the displacements in every celestial body. Whether the relative movements in the Earth's interior had such a zonal character or not, can be inferred only by comparison of the logical consequences of this assumption to the *actual* distribution of the principal features in the crust.

7.7. Formation of Principal Features of the Earth's Crust

It has been shown by the author (Jardetzky [1], [4], [5], a.o.) that the hypothesis of zonal rotation yields an explanation of many of the principal characteristics of the Earth's crust and does not contradict others for which complementary causes must be suggested. The complexity of deformations during geological history as well as the present shape of the crust require an agreement as to which facts are to be considered as the principal ones. By neglecting such a definition, much confusion has been produced

in the discussion of some geophysical theories, and, therefore, and attempt has been made to present a list of most important facts which should be taken into account in the first place (Jardetzky [14]). (a) The outer layer of the Earth differs but little from a spherical shell; the density is increasing with depth. (b) The complexity of the structure is characteristic to the uppermost layer (33 km. thick). A layer now termed the crust is petrologically separated from the rest of the solid part of the Earth which is called the mantle. Here we use the term crust in the sense of a solid shell. (c) The core should be characterized as liquid even at the present time according to seismic information. Its radius is 3400 km.

The hypothesis of zonal rotation is compatible with this group of data. From (a) we can draw the conclusion that at the time

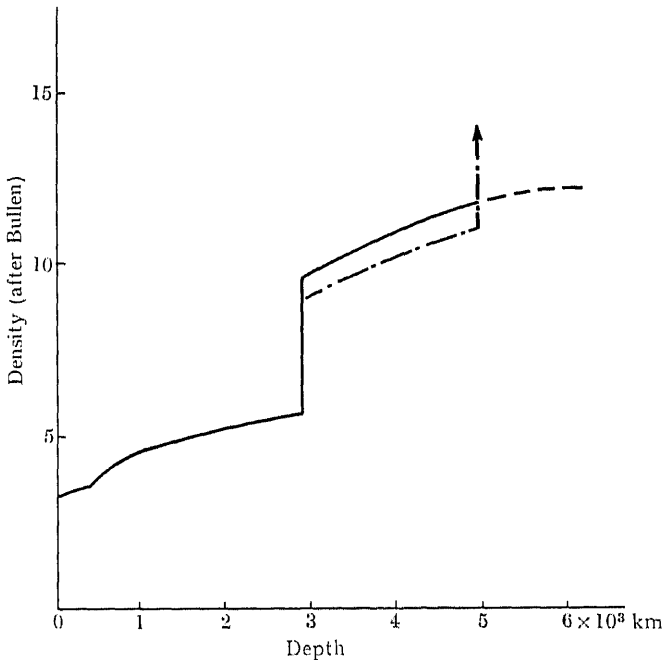


Figure 7

when the mantle became thick enough to resist any deformation on a large scale, the rotation of the Earth was almost a rigid-body rotation with a small angular velocity. Slow internal movements did not essentially change the stratification which differs but little from that in a heterogeneous figure of equilibrium (corresponding to the law of densities which is represented in Figure 7, Bullen [2]).

Obviously, the most essential effect of internal movements could be provided during the formation of the crust and eventually of the first thin solid layer in the mantle. The data concerning the core at present support in general the idea that parts of the crust and mantle were fluid during some geological periods. Now, the most important general facts concerning the crust are (d) the continents are mostly granitic, while the bottom of oceans is basaltic; (e) the actual distribution of land and sea for which probably one of the most striking characteristics is the existence of geographical homologies; (f) the changes in this distribution in different geological periods; (g) the formation and distribution of mountain chains; and (h) the phases in the formation of mountain chains.

The hypothesis of a zonal rotation yields a very simple explanation of the horizontal differentiation mentioned in (d). It shows also in which manner the Old World could be separated from the Western continents and dragged by the underflowing magma to the East. Australia, the East Indies, and many islands in Pacific could be broken away from Asia and Africa and also displaced to the East by the equatorial belt of fluid substratum accelerated in this direction. Thus, the hypothesis of a zonal rotation of the Earth's interior seems to be in very good agreement with principal characteristics of the actual distribution of land and sea. As to the changes in continental blocks and in ocean basins, a special investigation is necessary to estimate the effect of the magma currents during different geological periods. There is no doubt that some changes were due to other causes than the drag along the crust-mantle interface. The deformation of the whole Earth's body because of its displacement with respect to the axis of rotation could also be an important factor. Tides in a still fluid interior of the Earth when the crust was relatively thin as well as departures

from a simple law of zonal rotation (discussed by the author) probably affected the consolidation of the crust at many places. In the formation and distribution of mountain chains, however, the stresses produced by the zonal rotation of the magma again played the most important part. The Tertiary mountain belt, for example, has been easily explained by this hypothesis as well as some other mountain ranges (Jardetzky [5], [7], [8]). In order to understand the formation of remaining ranges of mountains (like the Appalachians) more data are needed to find the field of forces resulting in those folds. The last point mentioned above, namely, the cyclic character of the mountain building, i.e., the phases in the deformations can be explained by the interference of two factors: the increasing viscosity of the underlying magma and the advancing thickening of the crust. The second factor shows that larger stresses are necessary in later geological periods to produce the folds, while the first yields increasing horizontal stresses which can reach a value required for such a folding. However, it must be pointed out that without the hypothesis of the zonal rotation it cannot be seen so easily why the field of these forces existed exactly at the places which are determined by the actual distribution of mountain ranges.

CHAPTER VIII

Varying Figures

8.1. Small Oscillations

A set of figures of an isolated fluid mass the existence of which could be proved by different methods includes stable figures of equilibrium or figures of a fluid in a permanent zonal rotation as well as unstable configurations. The oscillations about a stable figure and progressive deformations represent two types of problems which help to elucidate the behavior of a celestial body during a more or less long time interval. More attention has been paid in celestial mechanics to the former problems. The periodic changes investigated were either the free oscillations of a fluid mass or small changes due to the action of external periodic forces.

Poincaré [1] studied the free oscillations of a liquid ellipsoid, but the question whether it is possible that all particles of a liquid mass have the same period of oscillations needed further discussion. It was shown by Appell in 1920 and Cartan [1] that the conclusions of Poincaré are correct despite the doubts voiced by Globa-Michailenko [1].

Hill found in 1884 that the integral of kinetic energy in case where the particles of a fluid mass have the velocity

$$(8.1.1) \quad \mathbf{v} = \nabla\varphi + m\nabla\psi$$

can be written in the form

$$(8.1.2) \quad fU - \int \frac{d\mathcal{P}}{\kappa} = \frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 - \frac{m^2}{2}(\nabla\psi)^2$$

This equation can be used now instead of (1.1.5).

On assuming that the level surfaces are determined as before by a parameter a , (8.1.2) takes the form

$$(8.1.3) \quad fU = \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 - \frac{m^2}{2}(\nabla \psi)^2 + \text{funct. } (a)$$

Thus, for example, the series expansion given by Liapounov (Chap. V) can be applied as long as these surfaces differ but little from ellipsoids and the equation can be solved for the function ζ which will determine the deformation. This would be possible, of course, if three functions φ , m , ψ were known. They must, however satisfy certain general conditions given by Clebsch [1], and the problem becomes much more difficult than problems concerning figures of equilibrium. Jardetzky [4] has calculated the first approximation for periodic changes of a meridian of an oscillating homogeneous ellipsoid using the equation (8.1.3), a given law of oscillation, and the method of Liapounov.

The periodic variations of a liquid ellipsoid such that at every instant the shape of the liquid is exactly ellipsoidal were determined by Dedekind [1]. (See also Lamb [1].)

It is well known that methods different from those discussed in preceding chapters were applied to the problem of small oscillations of a liquid mass under the action of periodic forces. The theory of tides represents a broad field, the results of which are not used in this investigation.

8.2. *Progressive Deformations*

The problem of progressive variations of a celestial body has first been met in a particular form. It has been shown, namely, that there are continuous sets of figures of equilibrium, and a natural conclusion was drawn from this fact. It has been assumed that during its evolution, a celestial body will be subjected to such changes that its free surface will coincide subsequently with figures belonging to one of these sets. It is evident that this is too idealized a case. In that aspect, in which this problem was posed in order to draw some conclusions about the evolution of stars, no essential progress has been made to our knowledge about the progressive deformation (see, for example, Jeans, [3]). There is no doubt that the case of an isolated homogeneous mass is of second-

ary importance. In most cases there are factors like action of other members in the solar system, which will change such an undisturbed evolution.

Even if we cannot expect to have more or less important applications of the theory of a varying figure in case of every celestial body, this theory should be more developed because of the deformations of the Earth.

We can see two classes of deformations in case of the Earth. To the first belong all relative displacements of the parts of the outer layer. There seems to be no disagreement now about the assumption that the folding of greater mountain ranges involves horizontal displacements to 100 km. (Jeffreys [1]). There arises only the question about the horizontal displacements of the order of several thousands kilometers, as it is required in the theory of continental displacements. All such variations in the mass distribution would affect the shape of a geoid, and this figure in Carboniferous was different from that determined at the present time.

The horizontal deformations in the crust as well as some epirogenic displacements could result from internal movements, for example, from the zonal rotation in the Earth as it was pointed out in Chapter VII.

A deformation of another type could be due to displacement of the axis of rotation in the body. The migration of the poles of the Earth has been discussed in geophysics and related sciences and according to one interpretation of geological data, the poles moved progressively over the surface of the Earth during all geological periods. One of the most striking phenomena in the past, the transgressions and regressions, changing the distribution of land and sea in many different ways could be at least partly explained by the displacement of the axis of rotation in the Earth. But, in general, the rotation of a deformable body is a problem which is still far from a complete solution. The basic equations of motion of a body which can change its shape during the motion were given by Liouville (see, for example, Tisserand [1]). Darwin [2] attempted to determine the influence of geological changes on the Earth's axis of rotation by assuming that the body is slightly deformable and

the changes are slow. Different conditions concerning this problem were investigated also by Picart [1]. Darwin and Schiaparelli introduced the hypothesis of continuous adaption of the body to a new figure of equilibrium which results from the displacement of the axis of rotation with respect to the axes of inertia, since from the equations of Liouville one cannot determine completely the motion of a deformable body. In general, as is well known, in such a body the concept of the axis of rotation must be introduced in a more precise manner and this can be done in different ways (see, for example, Bilimovitch [1]). The most useful seems to be the axis of rotation of an equivalent rigid body which is defined by the condition that it has the linear momentum and angular momentum equal to those of the given body (Jardetzky [4], [7], [8], [9]). We restrict ourselves by mentioning here a theorem of Milankovitch [1] which seems to be thus far the best approach to solving the problem of migration of the Earth's poles. We can consider the deformation of the entire body of the Earth by assuming it to be elastic, plastic, or in any state differing from the rigid. On the other hand, we can assume that a liquid (or plastic or fluidal) core is covered by a solid crust which may change its shape.

In both cases the asymmetric distribution of masses in the uppermost layer is the factor which plays the most important part in the displacement of poles. Let us consider the moments of inertia of this asymmetric shell. Every point at the free surface can be associated with the value of the moment (J) taken with respect to the central axis passing through this point. Then the free surface becomes a field of J . Now, it has been proved by Milankovitch that the progressive displacement \mathbf{v} of the Earth's pole occurs at each instant in the direction of the gradient of J , i.e.,

$$\mathbf{v} = k\nabla J$$

The progressive displacement of the pole is, of course, combined with the well-known periodic changes in the position of the pole (precession, nutation).

The general conclusions of this theory concerning the rotation of a deformable Earth can obviously be applied to every celestial

body in the period of its solidification. The most important of them are: (a) the progressive displacement of the crust over the core during the period of solidification is due to the simultaneous rotation about an axis in the equatorial plane and adaptation of the mass to a continuously changing field of gravity forces (i.e., there exists a component of the angular velocity in the equatorial plane of the body); (b) the primary cause of such displacement is the action of the moment of centrifugal forces arising from the asymmetrical distribution of masses in the crust; (c) the adaptation is possible if the crust is flexible and the core displays the properties of a liquid; and (d) the currents in the interior can play in general an important part in the sliding of the crust over the fluid interior, but in such cases as that of the Earth, their influence on sliding seems to be of secondary significance.

To get more insight into the character of these changes, small and slow as they are, many problems must yet be solved.

CHAPTER IX

Systems Composed of Fluid and Rigid Parts

9.1. Rigid Core and Fluid Envelope

After probably more than four billion years of its existence, the Earth is still composed of solid and fluid parts. If we do not take into consideration the hydrosphere and atmosphere, the most appropriate model of it is a solid body with a cavity filled with a liquid. Of course, to simplify it, it is assumed that the mantle and crust are rigid. However, in some theories the existence of a rigid core in a liquid Earth is admitted for a certain period of its history.²¹ The problem of a figure of such a system is reduced to that of stratification in the liquid part under given conditions. This problem has not yet been discussed in more detail. A short discussion of it is given in Poincaré's lectures [2] and in another investigation of Lichtenstein [5]. In the latter only, the general equations are set up to determine the exact form of the sea covering the Earth. No solution in terms of any definite functions is obtained.

Even, if we restrict ourselves to the figures of equilibrium of a system composed of fluid and rigid parts, it is easy to see that there will be a great variety of problems due to the fact that different assumptions can be made concerning the composition of each part as well as the shape of the rigid core or crust. The simplest problem of this kind is obviously the absolute equilibrium of a fluid layer overlying a spherical rigid core. One solution is evident, namely, a stratification in concentric spheres, if the pressure at the free surface is zero. It seems that there are no essential changes required in the proof of the uniqueness and stability of the figure of equilibrium given for the case where there is no

²¹ In a recent theory, the solid inner core is supposed to be separated from a solid mantle by the liquid part of the core.

rigid core. New complications arise, however, in the case of a relative equilibrium. For the fluid part of the system, the equations (1.1.1)–(1.1.8) hold under the condition that U represents the sum of potentials U_i and U_e of the fluid and rigid part respectively. The equation (1.1.5) must be written in the form

$$(9.1.1) \quad fU_i + fU_e + \frac{\omega^2 s^2}{2} = \int \frac{d\phi}{\pi} + \text{const.}$$

and this condition holds for the volume V_i of the fluid. At the free boundary of it the condition $\phi = 0$ or

$$(9.1.2) \quad fU_i + fU_e + \frac{\omega^2 s^2}{2} = \text{const.}$$

must be satisfied, but there are no such simple conditions for the inner boundary of V_i . Whether we can assume that the gravity center of the core coincides with the center of gravity of the whole system depends on the shape of the volume of the rigid part V_e and on the density distribution in it. Only under the simplest assumptions about the density distribution and the shape of the core, can one expect to obtain a solution. Suppose that the core is a homogeneous sphere and that both the core and the fluid layer are rotating with a constant angular velocity. Suppose also that the whole system is isolated in space. The permanent existence of a plane of symmetry, through the mass center of the whole system seems to be very plausible. Besides the symmetry with respect to this equatorial plane, the axial symmetry can also be expected. Maybe, it can be proved that figures of the type of Jacobi's ellipsoid are impossible in the case of a spherical core. However, a rigid ellipsoidal core with three unequal axes seems not to contradict the existence of such figures. For a uniform rotation of a mixed system (liquid and rigid or solid parts) a necessary condition is, as mentioned in section (1.2) that the axis of rotation coincides with one of the axes of inertia. This condition is obviously satisfied in case of symmetrical figures defined above. Moreover, the spherical rigid core once set in a uniform rotation of this kind will continue to rotate about an axis fixed in

space, since the resultant moment of pressures at its surface as well as that of forces of attractions will be zero. At its surface $r = r_0$ and the potential

$$(9.1.3) \quad U_0 = \frac{m_0}{r_0}$$

where m_0 is the mass of the core is constant. By (9.1.1) it is at $r = r_0$

$$(9.1.4) \quad fU_i + \frac{\omega^2 s^2}{2} = \int \frac{dp}{\kappa} + \text{const.}$$

but there is no reason to assert that the pressure (and density) have constant values along this surface. Thus, (9.1.4) becomes a new boundary condition which must be satisfied by a function which will determine the stratification.

If we want to apply the method of Liapounov to a problem of this kind, the attempt should be made first to further simplify the conditions. For example, let us consider a very slow rotation of a slightly heterogeneous liquid in a layer around a homogeneous spherical core. Then, the angular velocity being very small, the level surfaces F_a will probably differ but little from a set of concentric spheres E_a with the radius $a\sqrt{\rho}$. Let $a = 1$ be the value of parameter a corresponding to the free surface, i.e., to the sphere E . In the law of densities $\kappa = 1 + \delta\varphi(a)$ (4.1.1) the parameter δ has very small values again and the equations

$$(9.1.5) \quad \begin{aligned} x &= a(1 + \zeta)\sqrt{\rho} \sin \vartheta \cos \psi \\ y &= a(1 + \zeta)\sqrt{\rho} \sin \vartheta \sin \psi \\ z &= a(1 + \zeta)\sqrt{\rho} \cos \vartheta \end{aligned}$$

represent the level surfaces F_a . They differ from concentric spheres E_a and, therefore, one part of the set of F_a will intersect the boundary of the core. Thus, at different points $P(x, y, z)$ of the sphere $r = r_0$ unequal values of a should be taken in expressions (9.1.5). If the definition of the parameter a given in Section (4.1) is used, we have to assume now that the volumes of F_a and E_a are equal for $a \geq a_1$, where a_1 must be determined by a special condition. The volume of F_a is composed of the volume of the core $\frac{4}{3}\pi r_0^3$ and that of the liquid layer. Changing the notations for

axes in (4.1.4) and other formulae in section (4.1), we obtain the condition

$$\frac{4}{3}\pi r_c^3 + \frac{\varrho\sqrt{\varrho}}{3}(a^3 - a_1^3) \int (1 + \zeta)^3 d\sigma = \frac{4}{3}\pi a^3 \varrho\sqrt{\varrho}$$

Hence, if the condition (4.1.6) has to hold in this case as well, we must have $a_1^3 \varrho\sqrt{\varrho} = r_c^3$. The potential U_i in (9.1.1) could now be taken in the form (4.1.7) with the limits of the integral equal to a_1 and 1. This would make it difficult to apply Liapounov's method. We know, however, that the problem of figures of equilibrium in case of a liquid mass could be reduced to the equation (4.3.16) when the potential is represented by the sum (4.3.4). The first term of this sum (Ψ) corresponds to the constant density $\kappa = 1$. Since the potential due to attraction of a homogeneous body does not depend on its aggregate state, we could assume that Ψ corresponds to the homogeneous figure composed of the rigid core (its density being $\kappa_1 = 1$) and the liquid of equal density filling in the rest of the volume. The remaining part of the sum $U_i + U_c$ would be due to the excess of density $\delta\varphi(a)$ in the volume $V_{\text{total}} - V_{\text{core}}$. This would be equivalent to the modified law of densities in which we take $\varphi(a) \equiv 0$ for $a < a_1$, and $\kappa = 1 + \delta\varphi(a)$ for $a \geq a_1$. This condition will affect the term Φ in (4.3.4). Accordingly, we then have to assume that the function $\zeta = \zeta(a, \vartheta)$ which represents the deviation of level surfaces F_a from the spheres E_a , will be determined by the equation (4.3.16)

$$(9.1.6) \quad R\zeta - \frac{1}{4\pi a^2} \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = W + P(a)$$

in the interval $(a_1, 1)$, i.e., in the liquid part of the system. Now, in (4.3.17) and (4.3.18) the $\Phi(0) \equiv 0$ because of the vanishing of $\varphi(a)$ at the mass center. Therefore, the difference between this problem and that concerning a liquid mass (Chap. IV) will be expressed by new values of the function Φ . We can assume again that the function ζ is expanded in the series

$$(9.1.7) \quad \zeta = \sum_{i=1}^{\infty} \zeta_i \delta^i$$

and apply transformations similar to those in Section (4.3). Liapounov's results show how difficult it is to calculate effectively the functions ζ_i which yield a set of approximations to an exact solution. We will not make an attempt here to find even the first functions, since we do not expect that the solution of this problem will have any important applications.

In the theory of tides, the free surface of the ocean in the state of equilibrium on the solid body of the Earth is taken as the surface of reference to determine the oscillation with respect to it. M. Brillouin [1] pointed out, however, that according to his investigations, this surface is unstable. His conclusion is that there is no stable permanent configuration of a liquid which has no relative motion with respect to a solid having a uniform angular velocity.

We have mentioned certain very restrictive but necessary conditions of equilibrium of mixed systems, but the question of stability of such systems should be investigated in more detail.

9.2. Liquid Mass and Floating Rigid Bodies

The ratio of masses as well as that of densities are factors which determine whether a problem concerning the rotation of a system composed of liquid and rigid parts will be important in the theory of figures of celestial bodies or not. If, for example, the liquid mass is much larger than the mass of one or several rigid parts of a system and these parts are less dense, we can consider different configurations which will have this liquid mass with floating bodies. A period during which a planet or a former star can be treated as a large liquid mass differing but little from a figure of equilibrium and partly covered with floating bodies seems to be one of the natural links in the evolution of celestial bodies. An objection is sometimes made that those substances that are present in the outer shell of the Earth became more dense through solidification and, therefore, had to sink in the remaining liquid mass when the Earth was cooled at its free surface. Since the increase in density with depth is a well-known factor, the question arises as to how far these solidified parts of the outer layer can sink. The stratification in a celestial body even if it is in the gaseous state seems to be an-

other well-established factor. Can we postulate that the law of density distribution is invariable during the whole history of a celestial body and that such solidified parts of the outer layer would always sink to a large depth? This assumption is made implicitly in some investigations dealing with the solidification of the Earth. Let us assume that after the time when small solids were moving toward the center in the vertical direction, melting and mixing with lower layers, the differentiation could reach such a stage that no more deep sinking would be possible. Then, there could be a larger part of an outer layer solidified such that its mean density was less than that of underlying layers. In case of the Earth such an assumption is supported by the isostatic mass distribution in the crust. Even if imperfect, the isostasy speaks in favor of the concept of floating continental blocks on a still hot and liquid planet.

The solidification in the whole outer layer of a rotating liquid planet can occur simultaneously, and if no other factors will oppose this process, one can expect that the planet will be covered by a uniform crust. This is the viewpoint of many investigators who expect that the solidification to a great depth is a process progressing very rapidly. It could even be accomplished in a few-thousand years. On the other hand, the formation of a single continental block that floated on a liquid substratum and by bursting gave rise to present continents was the basic idea of Wegener's theory [1] of formation of continents. The existence of such a pangea or a protocontinent was not explained in this theory, but the author (Jardetzky [1], [4], [7], [8]) has shown that the hypothesis of a zonal rotation yields a simple reason for the formation of a single continent on a rotating liquid planet. Since many features of the crust can be created by the displacement and deformations of continents, as has been shown in Section (7.7), the motion of a body floating on a large liquid mass becomes an important problem. The existence of continents and their deformation affects, as it was pointed out earlier, the figure of a planet as a whole and is responsible for local departures from a more or less regular shape.

Unfortunately, because of the great complexity of the problem, there are but few attempts to treat it in mathematical form. Some remarks of general character can be found in investigations concerning this very remote part of the Earth's history, (W. Thomson [1] Vol. IV, p. 189; Jeffreys [1]). They deal, for example, with the question of stability if there are many small floating bodies collected on one side of the planet. An attempt to discuss and solve the problem in an exact form was made by the author (Jardetzky [4]), but despite many simplifications, it remained intractable. It is obvious that in solving problems of this type, an explanation is sought of those characteristics of figures of celestial bodies which are due to the relative displacements of their parts. Therefore, the figures of equilibrium of a liquid mass with floating bodies are of secondary interest. It can be shown that very special conditions must be satisfied to secure the existence of a figure of equilibrium of this kind. One of these conditions concerning the position of the axis of rotation with respect to the axes of inertia was mentioned in Section (1.2).

It is easy to give some examples where this condition will be satisfied. Spherical polar caps homogeneous and regular in shape placed on a denser (nonrotating) liquid mass could remain in these positions but probably not indefinitely. The problem of a nonrotating liquid planet with a floating continent has been investigated by Jardetzky [3], [6], but as pointed out, the dynamical problem is actually of more importance. Less precise methods than those discussed in the preceding chapters were used to get more insight into the phenomena connected with the behavior of a material system composed of liquid and solid parts. By such methods, the existence of the so-called "Polflucht"-force in Wegener's theory was analyzed. Several proofs were given in geophysics in order to show that, if there is a continent floating on a rotating liquid planet, a component of gravity force acting towards the equator must tend to displace this continent in this direction. Thus, if we assume the isostasy which is defined by the condition of floatation, such a component must appear because of the higher position of the floating body as compared with the

stratification in a totally liquid mass in equilibrium. That not all the conditions under which such a component will be directed toward the equator are always satisfied is apparent. The evaluation in case of the Earth yields a very small numerical value of this force.

Future investigations will clear up the conditions for the existence of horizontal as well as vertical forces acting on a floating continent. Perhaps even a rough solution will be sufficient in this case, but one has to keep in mind that this solution must satisfy the equations of motion of a liquid part and six equations for a rigid body. Of course, the chances of finding such a solution will be increased by the assumption that the motion of each part of the material systems differs but little from a pure rotation and that the configuration is almost a figure of equilibrium.

It is evident that a better approximation to the actual figure of solid celestial bodies must be obtained by taking into account viscosity and elasticity of bodies, the factors like the drag produced by internal movements, etc. Therefore, we should be able to estimate the role played by each factor in a more precise way than it has been done before. This is, however, a very difficult task. As an example, we can mention an investigation of Prey [2] concerning the theory of land connections between the existing continents. According to this theory, some parts of the crust have sunk in a viscous substratum and were covered by the ocean. Before that time they could represent connections between other parts of land. The reality of such process in some cases can hardly be doubted, but the mathematical analysis in the form given by Prey leads to some inconsistencies in the order of the magnitude of viscosity and in the time required for sinking. The problem discussed concerns the radial displacement of a circular continent floating on a viscous liquid planet; the inertial terms in the equations being neglected. The difficulties met in this problem may be easily understood, if one takes into account that to represent an island by means of a spherical function, terms up to the 16th order are required.

9.3. Rigid Bodies Having Cavities Filled with Liquids

A new type of problem concerning the motion of a system composed of liquid and rigid parts different from those discussed in preceding sections includes all cases where liquid fills the cavities in a rigid body. Mass or density ratios can be arbitrary in these problems.

A problem of this kind approximates the present state of many planets probably quite fairly, but it must be assumed that the figure of the system has been determined by the conditions corresponding to the time interval when deformations in the solid crust (or mantle) were still possible on a large scale. The adaption to a figure of equilibrium has been one of the most important among these deformations. The changes in the stratification in the core followed obviously the variations of the form of the crust, but all the deformations seem to have been small unless some catastrophe occurred. The fact that even the results of Clairaut's theory, which is nothing but a first approximation, agree fairly with the observations confirms this statement. On a large scale, the mass distribution in the Earth corresponds to that in a heterogeneous figure of equilibrium. In the crust, the isostatic layering is not quite perfect, but if, for example, the reduction for isostasy is taken into account when the ellipticity of the Earth is calculated from observation, the value agrees with Clairaut's value. The actual level surfaces of the Earth differ but little from the ellipsoids and even from a set of spheres. If certain corrections and a more precise theory are desired, the effect of different factors must be accounted for in further investigations. One of these factors is the probable variation of the angular velocity during the solidification of the mantle.

The flexibility of the crust as a whole and its plasticity are well known to geophysicists. The tendency of some parts of the continents to restore the isostatic layering disturbed by an excessive load during the last ice age is observed even at the present time and is a very important phenomenon. It proves the fact that the figures of planets must be better approximated under the assumption that their crusts are deformable.

Because of extreme complexity, one can only hope that in future investigations, solutions will be found in which use will be made of methods similar to those of Liapounov (see Chap. IV). The series expansions given for the potential of a body are not restricted to a heterogeneous liquid. It is evident that they depend on a given law of densities in a body and the fact that the stratification differs but little from an ellipsoidal one. The representation of the potential can be used in problems concerning rigid, solid, and plastic masses.

As to the motion which differs from a rotation about an axis fixed in space and in the body, the investigations of Joukowski [1], Volterra [1], and Stekloff [2] yield basic results. In these investigations, the shape of a rigid body and the form of a cavity filled with a homogeneous liquid are given, and the "deformations" are admitted only in the sense of internal movements.

CHAPTER X

Fluid Mass and Centers of Attraction

10.1 General Problem

The small changes in the figure of a rotating fluid mass due to the existence of some other bodies can also be treated by the method of Liapounov if the basic conditions required by this method are satisfied. In the problem of the figure of a liquid planet which is affected by the attraction of other members of a given "solar" system, the most precise method has been applied by Lichtenstein [2]. In earlier investigations, certain particular cases of the problem have been treated by other methods. The double-star problem in its simplest form when the dimensions of the second star are neglected, that is, it is assumed that this star acts as a simple mass center, or the tidal problem are the well-known examples.

The equations of motion of a rotating fluid mass attracted by several centers have the form (1.1.3) again, but the potential U is now the sum of the potential U' of the fluid mass itself and U'' due to the masses m_i located at $C_i(x_i, y_i, z_i)$. The integral of these equations which will yield the figures of the fluid mass is, therefore, (1.1.5) or

$$(10.1.1) \quad f(U' + U'') + \frac{\omega^2 s^2}{2} = \int \frac{dp}{\kappa} + \text{const.}$$

We assume that the motion of the fluid mass is a rigid-body type of rotation, but on changing the second term of the left-hand member, as has been done in Chap. VII, we can also account for a zonal rotation. Only the simpler case is to be considered here, but it must be pointed out in advance that even a single, far-located mass center will be able to change the state of a uniform rotation of either a perfect or viscous fluid. In the first approximation,

equation (10.1.1) holds for each instant. However, the term U'' can depend on time, and, in this case, the function ξ which will determine the departure of a level surface from an ellipsoid becomes a function of time. Moreover, in case of an isolated fluid, we could assume that the axis of rotation passes through the mass center of the whole mass. In the new problem, the mass center of the fluid is moving in general with respect to the mass center C_s of the whole system. It is obvious that different assumptions can be made about the order of magnitude of the masses involved as well as about the ratio of the largest dimensions in a fluid mass and the distances between the mass centers. Accordingly, different approximations must be used, and we shall mention few examples. In the double-star problem, the case of a uniform rotation of both components with respect to the axis passing through the mass center of the system had been considered. If the masses are m and m_1 and the x axis is directed toward the mass center m_1 , the distance between C_s and C (the mass center of the first component) will be $s_c = m'l/(m+m')$ where $l = CC_1$. Since in (10.1.1) s^2 is the distance from the axis of rotation, the second term in this equation takes the form

$$\frac{\omega^2}{2} \left[\left(x - \frac{m'}{m+m'} l \right)^2 + y^2 \right]$$

However, for celestial bodies, the periods of revolution and rotation are seldom equal, the angular velocity due to revolution being of a high order than that of rotation about the axis of the figure (exceptions: Mercury (?), Venus (?), Moon in the system Earth-Moon).

Let us assume now that the mass center of the whole system is in the close neighborhood of the mass center of the liquid mass (the figure of the Sun affected by planets or the figure of a liquid planet affected by the satellites). Then in (10.1.1) the distance s_c can be neglected. Another simplification concerns the expansion of the potential U_i due to the mass center m_i in a series. If l_i are distances of m_i from C (which coincides with C_s) and $l'_i = C_i P$, where P is a point of the fluid mass, we have

$$(10.1.2) \quad U'' = \sum U_i = \sum \frac{m_i}{l_i'}$$

Since

$$l_i'^2 = l_i^2 + r^2 - 2l_i r \cos \gamma_i$$

for the mass m_i , we obtain

$$(10.1.3) \quad U_i = \frac{m_i}{l_i} \left[1 - 2 \frac{r}{l_i} \cos \gamma_i + \frac{r^2}{l_i^2} \right]^{-\frac{1}{2}}$$

In spherical polar coordinates

$$(10.1.4) \quad \cos \gamma_i = \cos \vartheta \cos \vartheta_i + \sin \vartheta \sin \vartheta_i \cos (\psi - \psi_i)$$

and the last factor in (10.1.3) can be expanded in the series (see equation (1.4.13))

$$(10.1.5) \quad \sum_{n=0}^{\infty} \left(\frac{r}{l_i} \right)^n P_n(\cos \gamma_i)$$

where P_n are the Legendre polynoms and

$$\frac{r}{l_i} < 1$$

By (10.1.2)–(10.1.5) we write, if there are q mass centers

$$(10.1.6) \quad U'' = \sum_{i=1}^q \sum_{n=0}^{\infty} \frac{m_i r^n}{l_i^{n+1}} P_n(\cos \vartheta \cos \vartheta_i + \sin \vartheta \sin \vartheta_i \cos (\psi - \psi_i))$$

where (4.1.2)

$$(10.1.7) \quad r^2 = x^2 + y^2 + z^2 = a^2(1 + \zeta)^2 \{ (\varrho + 1) \sin^2 \vartheta \cos^2 \psi + (\varrho + q) \sin^2 \vartheta \sin^2 \psi + \varrho \cos^2 \vartheta \}$$

The form of the series expansion of the potential U' of the liquid mass is not affected by adding the term U'' in (10.1.1) and all transformations of Liapounov given in Chapter IV hold again. Therefore, the functional equation (10.1.1) can be reduced again to an integro-differential equation and subsequently to the set of integral equations of the type investigated in Chapters IV and VII.

This integro-differential equation is by (4.3.16)

$$(10.1.8) \quad R\zeta - \frac{1}{4\pi a^2} \int \frac{\bar{\zeta}' d\sigma'}{D(a, 1)} = W + P(a) + \frac{1}{4\pi a^2 \Delta} U''$$

where W and $P(a)$ are expressions (4.3.17) and (4.3.18) or the more general expressions given in Liapounov's theory for ellipsoids with three unequal axes. In order to obtain from (10.1.8) the integral equations yielding functions which can be used for calculation of different approximations, we have to determine some new parameters. In the problem of equilibrium of a single fluid mass, the parameter δ in the density law or λ in case of a zonal rotation have been considered as small quantities of the first order. In (10.1.6) however, the order of magnitude of three new factors: m_i , $1/l_i$, and r/l_i can be taken in an arbitrary manner. The first term of this series expansion is m_i/l_i and is, of course, in general variable. If we assume that the paths of masses m_i around the point C_s are circular or that the centers C_i are fixed in space, the ratio m_i/l_i is constant. It is small if either the masses m_i are small or the distances l_i large or both. This factor does not affect, however, the relative magnitude of terms of the sum (10.1.5) which depends on the ratio r/l_i . If we want to apply the method of Liapounov to the equation (10.1.8) we have to assume that the function ζ can be expanded in a multiple series of the form

$$(10.1.9) \quad \zeta = \sum \zeta_{gh\dots i} \delta_1^g \delta_2^h \dots \delta^i$$

where the small parameters δ_k are chosen in such a way that they can represent the effect of attraction of the mass m_i . To avoid unnecessary complications, δ_h and δ must be of the same order of magnitude and $\zeta_{00\dots 0} = 0$. Because of (10.1.7) from which it follows that r/l_i is of the order $\sqrt{\varrho + 1/l_i}$, $\sqrt{\varrho + 1}$ being the largest half axis of the free surface, it seems to be natural to assume that for the first approximation, the factor

$$(10.1.10) \quad \delta_i = \frac{m_i}{l_i} \frac{\sqrt{\varrho + 1}}{l_i}$$

is of the order of δ . The assumptions about the order of magnitude of three independent factors m_i , $1/l_i$, and r/l_i are combined in one,

but it is evident that this is not the only possible condition. Now, by substituting (10.1.9) in (10.1.8) the integral equations of the type (7.3.14) or (7.3.7) for the coefficients $\zeta_{gh\dots i}$ can be derived by transformations similar to those in case of a single parameter δ . To find effectively these coefficients very extensive calculations are required even for the first approximation

$$(10.1.11) \quad \zeta = \zeta_{10\dots 0}\delta_1 + \zeta_{01\dots 0}\delta_2 + \dots + \zeta_{00\dots 1}\delta$$

where the number of terms is $q + 1$, if there are q mass centers. The coefficients $\zeta_{00\dots 1} = \zeta_1$ is given by Liapounov (see Chap. VII) and the others can be calculated in the way used by the author in the problem of the zonal rotation of a homogeneous mass (Jardetzky [3]).

10.2. Recent Investigations

The classical problem of Roche concerning the figure of a homogeneous liquid mass subjected to the attraction of a far-removed mass center was recently generalized by Agostinelli [1]. He assumed that there is a set of such mass centers and that the changes in their positions can be neglected when a time interval is considered corresponding to several rotations of the liquid mass. All mass centers are supposed to be located in the equatorial plane of the liquid mass. It could be proved that under these conditions, a rigid body type of rotation is impossible. However, the free surface of the liquid can have the shape of an ellipsoid with three unequal axes, which do not change the magnitude or direction with respect to the centers of attraction. Then, the particles of the liquid move along elliptical paths.

The fundamental equation of the problem has the form

$$U_i + \frac{1}{2}\omega^2(x^2 + y^2) + C_1(x^2 - z^2) + C_2(y^2 - z^2) = \text{const.}$$

where the terms in C_i are due to the centers of attraction. Agostinelli [3] had also considered the case of a rotating heterogeneous liquid taking Roche's law for density distribution. To treat this case, he assumed that there is a complementary motion due to a dilation of the liquid. If the stratification differs but little from an ellipsoidal one, the potential can be expressed in terms of the polar and equatorial superficial ellipticities. This theory applied to the

Earth which is subjected to the lunisolar attraction yields for the equatorial ellipticity the limit 1.67×10^{-5} .

On making use of Poincaré's analysis, Agostinelli [3] found also that there are some critical ellipsoids which can give rise to new figures of equilibrium if slightly deformed. These configurations are obviously not figures of equilibrium in the strict sense, since the results hold under the assumption that the centers of attraction do not change their positions.

A generalization concerning the number of centers of attraction leads to another problem which presents a great interest. If we assume, namely, a very large number of such centers distributed in a ring concentric to a liquid mass, we obtain the problem of the ring of Saturn or of a ring-shaped nebula with a central star. The classical investigations concerning the first problem were discussed in the work of Appell ([1] Vol. IV) in more detail. We shall only mention a recent report by Nadile [1]. He could show, namely, that the free surface of the central liquid mass can take the form of an ellipsoid if the radius of the ring is very large and its cross section very small. This conclusion holds when the terms of an order higher than K^2 are neglected, K being the ratio of the maximum dimension of the central body to the radius of the ring.

CHAPTER XI

Figures of Compressible Masses

11.1. Stratification in Stars

The theories of figures of celestial bodies discussed in preceding chapters reached a very high level of mathematical perfection. There is no doubt they became a solid base for explanation of many features or properties of any kind of celestial bodies (solid, liquid, and gaseous). We have seen, however, in case of the zonal rotation, that the approximation based on equations of Mechanics only requires an improvement. We know, for example, that the equilibrium in a star is maintained by the gravity, the pressure gradient, and by the radiation and propagation of energy in it. From the mechanical viewpoint, the theories that should yield a better explanation of stratification in stars, must take into account the much higher compressibility of gases. On considering such a new factor, the investigators of figures of compressible masses made use of different methods.

Usually, it is assumed that stars and some planets or their atmospheres are composed of an ideal gas. The equation of state has the form $p v = R T$. There are two cases in which some problems can be also treated by using the theories mentioned in preceding chapters. For adiabatic processes, we have $p v^\gamma = \text{const.}$ and for polytropic variations it is $p = K \kappa^{1+(1/n)}$ (n = index of polytropy). Even if we have a so-called generalized polytropic gas $p = K \kappa^{1+1/(n(\kappa))}$ the relationship between the pressure and density is of the form $p = \varphi(\kappa)$, and the motion of the mass is barotropic. The fundamental equation of the theory of equilibrium and of a zonal rotation (1.1.5) holds for adiabatic as well as for polytropic states, and the stratification is determined by it. However, there is a difference in one point. We no longer can make use of the condi-

tion of equal volumes when deforming a figure of equilibrium, since in a compressible gas a change in the pressure field can produce an essential variation of density and volume. Thus, as a supplementary condition, equality of masses must replace the equality of volumes.

Instead of these equations in the problem of stratification in a gaseous celestial body, other conditions can also be used. e.g., for the absolute equilibrium, there are three equations for three variables: p , κ , U (assuming a spherical layering).

$$(11.1.1) \quad \frac{1}{\kappa} \frac{dp}{dr} = f \frac{dU}{dr}, \quad \nabla^2 U = -4\pi\kappa, \quad p = K\kappa^{1+1/(n(\kappa))}$$

The first equation is a special case of (1.1.2), the second is the Poisson equation. The substitution

$$(11.1.2) \quad \kappa = \theta^n, \quad \frac{n+1}{4\pi f} K = \alpha^2, \quad \xi = \frac{r}{\alpha}$$

transforms then Poisson's equation into the equation of Emden

$$(11.1.3) \quad \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

which yields a function $\theta = \theta(\xi)$, i.e., the law of densities $\kappa = \kappa(r)$ for a given $n(\kappa)$ and boundary conditions.

In the next approximation, as is well known, the pressure of radiation p^* is added in the first condition of equilibrium (11.1.1). Then, if m_r is the mass bounded by a sphere of radius r ,

$$(11.1.4) \quad \frac{1}{\kappa} \frac{dP}{dr} = \frac{1}{\kappa} \frac{d}{dr} (p + p^*) = -f \frac{m_r}{r^2}$$

This is equivalent to the assumption that an additional force, $\text{grad } p^*$, is introduced in the equation (1.1.2). It is not our purpose here to discuss all the hypotheses about the generation of energy and its propagation in the gas and absorption, which would make it possible to express p^* in terms of r . Different models were considered in Astrophysics. Each definite form of $p^*(r)$ is connected with certain assumptions about physical processes involved

and approximates more or less the actual state in stars. The nuclear processes are at the present time the most important point in this theory. The problem of stratification in stars in this form is not a question of how to determine the shape of level surfaces or surfaces of equal density or pressure. This shape is given and the density (or pressure) is to be found.

Further investigation of the stratification meets a new difficulty in the following fact. If in the equation of state a new variable, as, for example, the temperature T is introduced, the equations of Hydromechanics do not represent a more complete system of conditions that will determine all the variables (T included). It has been pointed out by Duhem [1] that the required sixth equation will be given by Thermodynamics. Nevertheless, as far as is known to the author, no general use has been made of this suggestion. Instead of this, several general conclusions were drawn directly from the equations of Hydrodynamics. If κ is not a function of a single variable p , the right-hand member in (11.1.4) is not an exact differential and the equation (11.1.5) cannot be obtained even if ω is constant. The three sets of surfaces $\kappa = \text{const.}$, $p = \text{const.}$, and $U + Q = \text{const.}$ now no longer coincide, and the term stratification loses its exact meaning. The motion of such a mass is called barocline, but it must always be specified what is meant by stratification. Some useful relationships can readily be obtained. Writing the equations (1.1.3), for example, for cylindrical coordinates and assuming the rotational symmetry (which is proved to exist), we have the first equation in the form

$$-\omega^2 s = f \frac{\partial U}{\partial s} - \frac{1}{\kappa} \frac{\partial p}{\partial s}$$

Eliminating p from this equation and the third (1.1.3), the condition for ω is

$$(11.1.5) \quad \kappa \omega^2 = \frac{f}{s} \int \left(\frac{\partial \kappa}{\partial s} \frac{\partial U}{\partial z} - \frac{\partial \kappa}{\partial z} \frac{\partial U}{\partial s} \right) dz + \text{funct. } (s)$$

(Pizzetti [1]). If we eliminate U from (1.1.3) it is, for example,

$$x \frac{\partial \omega^2}{\partial z} = \frac{\partial p}{\partial x} \frac{\partial}{\partial z} \left(\frac{1}{\kappa} \right) - \frac{\partial p}{\partial z} \frac{\partial}{\partial x} \left(\frac{1}{\kappa} \right)$$

This and two other equations show that

$$(11.1.6) \quad \text{grad}_z \omega^2 \times \mathbf{r} = \nabla \frac{1}{\kappa} \times \nabla p$$

where grad_z is the partial gradient in the z direction. Thus, the isobaric and isosteric surfaces coincide only if $\partial \omega / \partial z = 0$. The condition (11.1.5) shows that in general $\omega = \omega(s^2, z^2)$, if the symmetry with respect to the equatorial plane is taken into account. The conclusion drawn from the condition (11.1.6) seems to be rather surprising. The fact that isobaric and isosteric surfaces do not coincide in a fluid mass is due to a special form of a physical law concerning the property of this mass, namely, to the equation of state, for example, $p = p(\kappa, T)$. As interpreted in many investigations, a rigid body rotation of such a mass is a condition sufficient to make these two sets of surfaces coincide again, without any assumption about the temperature. Of course, there will be no contradiction, if isothermic surfaces correspond to the temperature distribution, say, $T = \text{const.}$ on the surface $\kappa = \text{const.}$ The surfaces $p = \text{const.}$ would belong to the same family, and each of the variables could be expressed in terms of a parameter like the parameter a of Liapounov: $\kappa = \kappa(a)$, $p = p(a)$, $T = T(a)$. Then, the rotation would be barotropic. It seems to be necessary, however, in order to avoid a contradiction in case of barocline rotation, to take care of the fact that such zonal movements represent only one part of a general circulation, the second component of which consists in meridional currents. In theories concerning the stratification in gaseous stars, it is usually assumed that the actual state is a convective equilibrium. The rising warmer masses expand adiabatically, the sinking are adiabatically compressed, but according to the assumption just mentioned, the shape of layers $\kappa = \text{const.}$, $p = \text{const.}$ is not violated by these currents. If this is a precise description, the components of acceleration due to these radial movements should be included in the fundamental

equations, even if we neglect certain differences in motion along a parallel. Thus, (11.1.3) will take the form

$$(11.1.7) \quad \begin{aligned} -\omega^2 x + a_x &= f \frac{\partial U}{\partial x} - \frac{1}{\kappa} \frac{\partial p}{\partial x}, & -\omega^2 y + a_y &= f \frac{\partial U}{\partial y} - \frac{1}{\kappa} \frac{\partial p}{\partial y} \\ a_z &= f \frac{\partial U}{\partial z} - \frac{1}{\kappa} \frac{\partial p}{\partial z} \end{aligned}$$

or in case of rotational symmetry

$$(11.1.7') \quad \begin{aligned} -\omega^2 s + a_s &= f \frac{\partial U}{\partial s} - \frac{1}{\kappa} \frac{\partial p}{\partial s}, & 0 &= f \frac{\partial U}{\partial \lambda} - \frac{1}{\kappa} \frac{\partial p}{\partial \lambda}, \\ a_z &= f \frac{\partial U}{\partial z} - \frac{1}{\kappa} \frac{\partial p}{\partial z} \end{aligned}$$

λ being the longitude, if we eliminate now U from (11.1.7) we obtain

$$(11.1.8) \quad \begin{aligned} \frac{\partial}{\partial y} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial z} - \frac{\partial}{\partial z} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial y} &= -y \frac{\partial \omega^2}{\partial z} + \frac{\partial a_y}{\partial z} - \frac{\partial a_z}{\partial y} \\ \frac{\partial}{\partial z} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial x} - \frac{\partial}{\partial x} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial z} &= x \frac{\partial \omega^2}{\partial z} + \frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \\ \frac{\partial}{\partial x} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial y} - \frac{\partial}{\partial y} \left(\frac{1}{\kappa} \right) \frac{\partial p}{\partial x} &= \frac{\partial a_x}{\partial y} - \frac{\partial a_y}{\partial x} \end{aligned}$$

These conditions will coincide with (11.1.6) only if

$$(11.1.9) \quad \text{curl } \mathbf{a} = \nabla \times \mathbf{a} = 0$$

In general, $\text{curl } \mathbf{a} \neq 0$ and instead of (11.1.6) we have

$$(11.1.10) \quad \nabla \frac{1}{\kappa} \times \nabla p = \text{grad}_x \omega^2 \times \mathbf{r} + \text{curl } \mathbf{a}$$

The vanishing of $\text{grad}_x \omega^2$ now would not lead to the coincidence of isobaric and isosteric surfaces. This conclusion also holds for viscous liquids.

We mentioned the method of Bondi and Lyttleton for the investigation of internal movements in a viscous liquid enclosed in a rigid core (Chapter IX). To discuss the problem of such internal movements in a gaseous star, one must also take into account the

fact that its stratification and the boundary surface are unknown. For compressible masses certain approximations have been made only for the case of small deformations of polytropic spheres due to a very slow rigid-body rotation. The assumptions are made, namely, that

$$(11.1.11) \quad p = K\kappa^{1+(1/n)}, \quad \kappa = \kappa(r, \vartheta) = \theta^n, \quad n = \text{const.}$$

If we put $\cos \vartheta = \bar{\mu}$ and make use of the substitution represented by second and third equation (11.1.2), the density $\kappa = \kappa(\xi, \bar{\mu})$. Since the stratification, i.e., the surfaces $\kappa = \text{const.}$, is changing when the angular velocity ω is varying, it is assumed that for small values of ω the density can be expanded in a power series

$$(11.1.12) \quad \kappa = \kappa_0 + \kappa_1 \frac{\omega^2}{2\pi f} + \kappa_2 \frac{\omega^4}{4\pi^2 f^2} + \dots$$

Suppose that in the first approximation on neglecting ω^4 and higher powers of ω we can take

$$(11.1.13) \quad \theta^n \approx \theta^n + \psi \frac{\omega^2}{2\pi f} n\theta^{n-1}$$

where θ is the solution of (11.1.3), $\kappa_1 = n\theta^{n-1}\psi$ and $\psi = \psi(\xi, \bar{\mu})$. If U and p are eliminated from (11.1.1) an equation for the function ψ will result. This equation can be written in the form (Milne [1], Chandrasekhar [1] and other papers)

$$(11.1.14) \quad \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \psi}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \bar{\mu}} \left[(1 - \bar{\mu}^2) \frac{\partial \psi}{\partial \bar{\mu}} \right] = -n\theta^{n-1}\psi + 1$$

if we make use of the expression for ∇^2 in spherical coordinates. A solution $\psi = \psi(\xi, \bar{\mu})$ of this equation will determine the stratification (11.1.12). The free surface that corresponds to the boundary condition $\kappa = \theta^n = 0$ is a spheroid. To solve the equation (11.1.14) Chandrasekhar assumed that ψ can be represented by the series

$$(11.1.15) \quad \psi = \psi_0(\xi) + \sum_{i=1}^{\infty} A_i \psi_i(\xi) P_i(\bar{\mu})$$

where P_i are Legendre polynomials. On inserting (11.1.15) in

(11.1.14), a set of equations is obtained to calculate the functions $\psi_i(\xi)$. In order to find the coefficients A_i Chandrasekhar introduced two new conditions. He puts equal expressions for the interior and exterior potential. The first derivatives of potential are also assumed to be equal at the boundary of the first Emden sphere $\xi = \xi_1$ which should be the figure of a nonrotating mass. Under these conditions, Chandrasekhar's solution is

$$(11.1.16) \quad \theta = \theta(\xi) + \frac{\omega^2}{2\pi f} \left[\psi_0(\xi) - \frac{5}{6} \frac{\xi_1^2}{3\psi_2(\xi_1) + \xi_1 \psi_2'(\xi_1)} \psi_2(\xi_1) P_2(\bar{\mu}) \right]$$

It has been pointed out by Krat [1] that the conditions introduced by Chandrasekhar can cause serious errors, since the action of masses exterior to the sphere $\xi = \xi_1$ seems to be neglected. Moreover, the question arises, whether both series expansions are convergent on this sphere or not. The remark of Poisson about a spherical layer (1.4.15) in which the expressions (1.4.13) given by Laplace are not valid could hold in the case under consideration too. Attempts were made to close the gap between those two expressions, but the question does not seem to be clear enough. The convergence of the basic series (11.1.12) in this theory of figures of compressible masses is just assumed and not proved subsequently to be correct. Krat [1] suggested a generalization of Chandrasekhar's results for the case of a zonal rotation

$$\omega^2 = c_0 \omega_0^2 + \sum_{i=1}^{\infty} c_i \omega_i^2 P_i(\bar{\mu}), \quad \omega_0 = \omega_0(\xi), \quad \omega_i = \omega_i(\xi)$$

In a previous investigation (Krat [1]) he had pointed out that if a given law of rotation of a polytropic gaseous mass is incompatible with conditions of its equilibrium, this will hold for any gaseous mass (generalized polytrop). As an example, the law $\omega^2 = \omega_1^2 \pm \omega_2^2 s^2$ is given which does not satisfy the conditions of equilibrium of a polytropic mass (despite the fact that the integral of (11.1.7') for $a_s = a_z = 0$ does exist).

For compressible gases in stars, a barocline motion seems to be the only possibility. Therefore, it will be very important to find precise methods which will enable us to determine all three sets of

surfaces, namely, $\kappa = \text{const.}$, $p = \text{const.}$, and $Q + fU = \text{const.}$ with any desired approximation.

11.2. Remarks on The Double-Star Problem

In its general form, this problem presents such difficulties that no method could yet be given to obtain its solution. We have seen in the preceding section some simplifications used in the case of a slow rotating isolated mass of a compressible gas. The difficulties arise first because of the fact that the components of a double star are compressible and heterogeneous. Besides the revolution about a common mass center, the rotation of both stars with different angular velocities is an important factor and the tides may dominate all other phenomena in a close system. Darwin considered the case of a rigid body rotation of the whole system. As a first approximation for homogeneous incompressible components, the ellipsoidal configurations were determined. However, one can expect that in a gaseous star the density is essentially increasing from the free surface towards the center. Therefore, the larger part of the mass is distributed in the "core," and the actual configuration will differ from an ellipsoidal mass. Bearing this in mind, Roche's model gains in its importance. For a double star, the potential due to centrifugal forces has to be combined with gravity potentials of two mass points. This model corresponds then actually to the restricted three-bodies problem. A set of equipotential surfaces being determined from it, certain indications about the paths of particles in stellar atmospheres have been obtained, and the discharge of the stream through the "critical" surfaces have been made probable. But the stars differ very much from Roche's model, and better approximations are required.

For polytropic stars, two particular problems became important. The tidal problem which is reduced to a rectilinear motion of the first component in the direction towards the second has been investigated by Chandrasekhar [1]. For the solution, series expansions similar to those mentioned in the preceding section were used and the coefficients A_i determined in terms of a parameter ν equal

to the ratio of the radius of an undisturbed configuration to the distance between the mass centers of star components. It could be shown that up to ν^5 , the deformations due to such tides can be superposed on those arising because of the adjustment of a double star to a shape corresponding to a rigid-body rotation. The figures of stars are approximated by ellipsoids if the terms of the order ν^4 and ν^5 are neglected. Further theoretical progress was made by Chandrasekhar [1], Krat [1], Russel [1], Cowling [1], and Walter [1].

The double-star problem has been treated by a different method by Lichtenstein [1]. Upon giving a far more precise formulation of the problem, only the general conditions were given in a mathematical form.

Liapounov's method can also be applied to the problem of a double star, and we shall write the general equations. If we assume that the densities of both components are functions of pressure $\kappa_1 = f_1(p)$, $\kappa_2 = f_2(p)$, the equations of motion of fluids in a double star are

$$(11.2.1) \quad \frac{d^2 \mathbf{r}_i}{dt^2} = \text{grad} \left(fU_i - \int \frac{dp}{\kappa_i} \right) \quad i = 1, 2$$

because of (1.1.2). Hence on denoting by $-Q_i$ the potential of accelerations

$$fU_i + Q_i = \int \frac{dp}{\kappa_i} + \text{const.}$$

on a level surface in each component. Since there are only two masses involved, the potential has the form

$$(11.2.3) \quad U_i = \bar{U} + \bar{\bar{U}}$$

i.e., it is the sum of potentials due to two components of a double star. In order to make use of the series expansions of Liapounov, certain conditions must be satisfied. The basic series expansion (4.1.15)–(4.1.16) holds for small (not infinitely small) values of the function ζ which determines the distortion of an ellipsoidal stratification. This can occur either for mean variations of density from the center to the free surface of each component, if there are

no restrictions about the magnitudes of axes of ellipsoids of $E_1(\sqrt{q_1+1}, \sqrt{q_1+q_1}, \sqrt{q_1})$ and E_2 respectively (see Section (4.1)) or, if the rotation of each component is very slow. In the second case, the ellipsoids E_1 and E_2 approach spherical shape and even for a very large increase in density from the free surface to the center, the level surfaces can be spheroids, i.e., ζ is very small. In both cases the distance between the mass centers of the components must be large enough in order not to produce an essential distortion in the stratification of each. Under these conditions, we can write (4.1.15)–(4.1.16) for each component, i.e.,

$$(11.2.4) \quad \bar{U} = \sum \bar{U}_i, \quad \bar{\bar{U}} = \sum \bar{\bar{U}}_i$$

where the terms \bar{U}_i are of the order ζ_1^i and $\bar{\bar{U}}_i$ of the order ζ_2^i . ζ_1 and ζ_2 being functions which in (4.1.2) will determine the stratifications of the first and second star, respectively. The functions U will be replaced by V and Φ , Section (4.3), if the laws of densities are given in the form

$$(11.2.5) \quad \kappa_i = 1 + \delta_i \varphi_i(a_i)$$

To simplify the problem one could assume that $\delta_1 = \delta_2 = \delta$ but the functions φ_1 and φ_2 are different and the parameters a_1 and a_2 are to be defined in general by two different conditions. If we take the definition of a_i as given in Section (4.1), these conditions are the equations (4.1.6) of Liapounov written for each component, i.e.,

$$(11.2.6) \quad \int \zeta_i d\sigma = - \int \zeta_i^2 d\sigma - \frac{1}{3} \int \zeta_i^3 d\sigma$$

If we consider the stars, equality of masses must replace the equality of volumes enclosed by F_a and E_a as mentioned above (see Krat [1], p. 148).

On writing the volume element (4.1.4) in the form

$$(11.2.7) \quad dV = \frac{4}{3} \frac{\partial}{\partial a} [a^3(1 + \zeta)^3] da d\sigma$$

and putting in this expression $\zeta = 0$ for the ellipsoid E_a we obtain, because of the new definition of the parameter a ,

$$(11.2.8) \quad \int_0^a \int_{\Sigma} \kappa dV = \Delta \int_0^a \kappa \int_{\Sigma} a^2 da d\sigma = 4\pi\Delta \int_0^a \kappa a^2 da$$

If we substitute the expression (11.2.5) the term equal to unity gives rise obviously to all terms in (4.1.6). The correction due to the second term in (11.2.5) can be easily computed, and we have two conditions for functions ζ_i as follows:

$$(11.2.9) \quad \int \zeta_i d\sigma = - \int \zeta_i^2 d\sigma - \frac{1}{3} \int \zeta_i^3 d\sigma \\ + \frac{\delta_i}{a_i^3} \int_0^{a_i} \varphi(a_i) da_i \int \frac{\partial}{\partial a_i} [a_i^3 (\zeta_i + \zeta_i^2 + \frac{1}{3} \zeta_i^3)] d\sigma$$

This equation must be used instead of (11.2.6) when the functional equations (11.2.2) will be transformed in integro-differential ones and subsequently into integral equations.

The acceleration potential Q_i may present an extreme complexity in more general cases if it does exist. Some approximate expressions for it can be written only by neglecting certain parts of acceleration. By assuming that invariable configurations can exist, a rigid body rotation of the whole system yields such a simple expression

$$(11.2.10) \quad Q_i = \frac{\omega^2 s_i^2}{2}$$

where s_i denotes the distance of a particle from the axis of rotation passing through the mass center of the double star. However, the motion of mass centers will not strictly be a Kepler motion. If this approximation is made, it may be put that

$$(11.2.11) \quad \omega^2 = f \frac{m_1 + m_2}{l_0^3}$$

where m_1 and m_2 are masses of the double-star components and l_0 the distance between their centers. This distance is the factor which affects the approximation in equations (11.2.2) and is used for the definition of the parameter ν mentioned above. It is evident that the ratio of the largest linear dimension in a component to the distance l_0 will determine the limit term to be used

in an approximation. In a model with homogeneous components, ν is the only parameter to be used even for close binaries. The relative order of parameters δ_i and ν has to be determined for heterogeneous components, since to transform the equations (11.2.2) we must use the series expansions of the form

$$(11.2.12) \quad \zeta_i = \sum \zeta_{i, gk} \delta_i^g \nu^k$$

and combine the terms of the same order.

Two equations for functions ζ_1 and ζ_2

$$(11.2.13) \quad \begin{aligned} f\bar{U} + f\bar{\bar{U}} + Q_1 &= \int \frac{dp}{\kappa_1} + \text{const.} && \text{for } S_1 \\ f\bar{U} + f\bar{\bar{U}} + Q_2 &= \int \frac{dp}{\kappa_2} + \text{const.} && \text{for } S_2 \end{aligned}$$

are of course very simplified in Roche's model in which the potential of another component is reduced to that of a mass point, and the fluid is considered as a compressible homogeneous liquid. Then, we have

$$(11.2.14) \quad \begin{aligned} f\bar{U} + f\frac{m_2}{l'} + Q_1 &= \frac{p}{\kappa_1} + \text{const.} && \text{for } S_1 \\ f\frac{m_1}{l'} + f\bar{\bar{U}} + Q_2 &= \frac{p}{\kappa_2} + \text{const.} && \text{for } S_2 \end{aligned}$$

Usually the figures are approximated by ellipsoids, but a progress can be made on substituting in (11.2.14) Liapounov's series expansion for \bar{U} in terms of ζ_1 or for $\bar{\bar{U}}$ in terms of ζ_2 . In case of (11.2.14) the functions ζ_1 and ζ_2 satisfy two separate equations, but even in this simplest case the solution of an equation of the form (11.2.140) will require very long calculations. Therefore, we restrict ourselves now to the last remark which concerns the character of condensation in a star as given by the law (11.2.5). It may be written in a form representing any degree of condensation if the parameter δ is not necessarily a small number. For the law $\kappa = 1 + \delta(H_1 - H_2 a^{2n})$ and any $n > 0$ limits for the density are determined at the center and at the free surface. They are $\kappa_0 = 1 + \delta H_1$, $\kappa_{f.s.} = 1 + \delta H_1 - \delta H_2$, but since $0 < a < 1$, the

larger is the value of n ; the slower variation of density occurs in the "core" of the star and the more rapid decrease in the "mantle." Maybe this law of densities would better represent the actual density distribution in a star and would not increase too much the mathematical difficulties of the solution.

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